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# Sequence Spaces of Operators on $l_2$

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# 1 Introduction

It is well-known that the Banach space  $\mathcal{B}(l_2)$  of all bounded linear operators on  $l_2$  is a non-commutative  $C^*$ -algebra with identity under the operator norm, the usual operator multiplication and the involution  $A \mapsto A^*$ . Furthermore, it can be regarded as the space of matrix representations of all bounded linear operators on  $l_2$  with respect to the standard orthonormal basis of  $l_2$ , which is exactly the space of all infinite matrices defining a linear operator on  $l_2$ .

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The *Schur product* (also known as the *Hadamard product* or *entry-wise product*) of two scalar matrices  $A = [a_{jk}]$  and  $B = [b_{jk}]$  of the same size is the matrix  $A \bullet B := [a_{jk}b_{jk}]$ .

In 1911, Schur proved that  $\mathcal{B}(l_2)$  is a commutative Banach algebra (without identity) under the operator norm and the Schur product operation. And it is not a  $C^*$ -algebra under the involution  $A \mapsto \overline{A}$ .

In 2007, Rakbud and Chaisuriya studied the duality and reflexivity of the following sequence spaces over any commutative  $C^*$ -algebra with identity:

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$$l_2(C(X)) = \left\{ \{f_k\}_{k=1}^{\infty} \subset C(X) : \left\{ \sum_{k=1}^n |f_k|^2 \right\}_{n=1}^{\infty} \text{ converges in } C(X) \right\},$$

$$l_2^b(C(X)) = \left\{ \{f_k\}_{k=1}^{\infty} \subset C(X) : \left\{ \sum_{k=1}^n |f_k|^2 \right\}_{n=1}^{\infty} \text{ is bounded in } C(X) \right\},$$

where  $C(X)$  is the  $C^*$ -algebra of all continuous functions on a compact Hausdorff space  $X$ , under the norm defined by

$$\|\{f_k\}_{k=1}^{\infty}\| := \left( \sup_n \left\| \sum_{k=1}^n |f_k|^2 \right\| \right)^{1/2} = \sup_{t \in X} \left( \sum_{k=1}^{\infty} |f_k(t)|^2 \right)^{1/2}$$

These spaces were first defined by Livshits, Ong and Wang in 2004.

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This result was straightforwardly generalized by Leelahanon and Chaisuriya in 2008 to any  $C^*$ -algebra with identity, in particular, the  $C^*$ -algebra  $\mathcal{B}(l_2)$ .

We can define analogously two sets of sequences in  $\mathcal{B}(l_2)$  via the Schur product as follows:

$$\mathcal{O}_b = \left\{ \{A_k\}_{k=1}^{\infty} \subset \mathcal{B}(l_2) : \left\{ \sum_{k=1}^n \overline{A_k} \bullet A_k \right\}_{n=1}^{\infty} \text{ is bounded in } \mathcal{B}(l_2) \right\};$$

$$\mathcal{O}_c = \left\{ \{A_k\}_{k=1}^{\infty} \subset \mathcal{B}(l_2) : \left\{ \sum_{k=1}^n \overline{A_k} \bullet A_k \right\}_{n=1}^{\infty} \text{ converges in } \mathcal{B}(l_2) \right\}.$$

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We expected to have that the set  $\mathcal{O}_b$  equipped with the norm

$$\|\{A_k\}_{k=1}^\infty\| := \sup_n \left\| \sum_{k=1}^n \overline{A_k} \bullet A_k \right\|^{1/2} .$$

is a Banach space. Unfortunately, we found that the function  $\|\cdot\|$  cannot be a complete norm on  $\mathcal{O}_b$ .

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## 2 Sequence Spaces over the Absolute Schur Algebras

In 1999, Chaisuriya and Ong define, for  $1 \leq r < \infty$ , the class  $\mathcal{S}^r$  of matrices  $A = [a_{jk}]$  such that *the absolute Schur  $r$ -th-power*  $A^{[r]} := [|a_{jk}|^r]$  defines a linear operator on  $l_2$ . And they proved that it is a Banach algebra under *the absolute Schur  $r$ -norm* defined by  $\|A\|_r = \|A^{[r]}\|^{1/r}$  and the Schur product multiplication. The authors also gave a nice relationship between the class  $\mathcal{B}(l_2)$  of all bounded operators and the algebra  $\mathcal{S}^2$ . They found that  $\mathcal{S}^2$  contains  $\mathcal{B}(l_2)$  as a non-closed ideal.

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From the pleasant structure of the Schur algebra  $\mathcal{S}^2$ , an interesting idea to study operators on  $l_2$  sequentially arises.

The set  $\mathcal{O}_b$  and  $\mathcal{O}_c$  (as well as the norm  $\|\cdot\|$ ) can be extended to be over the Schur algebra  $\mathcal{S}^2$  as follows:

$$\mathcal{O}_b = \left\{ \{A_k\}_{k=1}^{\infty} \subset \mathcal{S}^2 : \left\{ \sum_{k=1}^n A_k^{[2]} \right\}_{n=1}^{\infty} \text{ is bounded in } \mathcal{B}(l_2) \right\};$$

$$\mathcal{O}_c = \left\{ \{A_k\}_{k=1}^{\infty} \subset \mathcal{S}^2 : \left\{ \sum_{k=1}^n A_k^{[2]} \right\}_{n=1}^{\infty} \text{ converges in } \mathcal{B}(l_2) \right\}.$$

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### 3 Some Basic Properties of the Sequence Spaces $\mathcal{O}_b$ and $\mathcal{O}_c$

**Proposition 3.1**  $\mathcal{O}_c \subsetneq \mathcal{O}_b$ .

In the following theorem, we give a simple characterization of the set  $\mathcal{O}_b$ .

**Theorem 3.2** *A sequence  $\left\{ \left[ a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty}$  in  $\mathcal{S}^2$  belongs to  $\mathcal{O}_b$  if and only if the matrix  $\left[ \sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^2 \right]$  belongs to  $\mathcal{B}(l_2)$ .*

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**Theorem 3.3** (Cauchy-Schwarz's inequality) *For any two  $n$ -tuples  $(A_1, A_2, \dots, A_n)$  and  $(B_1, B_2, \dots, B_n)$  of matrices in  $\mathcal{S}^2$ ,*

$$\left\| \sum_{k=1}^n |A_k \bullet B_k| \right\| \leq \left\| \sum_{k=1}^n A_k^{[2]} \right\|^{1/2} \left\| \sum_{k=1}^n B_k^{[2]} \right\|^{1/2} .$$

From the Cauchy-Schwarz's inequality, the following corresponding Minkowski's inequality is obtained.

**Theorem 3.4** (Minkowski's inequality) *For any two  $n$ -tuples  $(A_1, A_2, \dots, A_n)$  and  $(B_1, B_2, \dots, B_n)$  of matrices in  $\mathcal{S}^2$ ,*

$$\left\| \sum_{k=1}^n (A_k + B_k)^{[2]} \right\|^{1/2} \leq \left\| \sum_{k=1}^n A_k^{[2]} \right\|^{1/2} + \left\| \sum_{k=1}^n B_k^{[2]} \right\|^{1/2} .$$

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In the space  $\mathcal{O}_c$ , Cauchy-Schwarz's inequality and Minkowski's inequality can be extended by the continuity of the operator norm on  $\mathcal{B}(l_2)$  to infinite sums as follows.

**Theorem 3.5** *For any  $\{A_k\}_{k=1}^{\infty}$  and  $\{B_k\}_{k=1}^{\infty}$  in  $\mathcal{O}_c$ ,*

$$(1) \left\| \sum_{k=1}^{\infty} |A_k \bullet B_k| \right\| \leq \left\| \sum_{k=1}^{\infty} A_k^{[2]} \right\|^{1/2} \left\| \sum_{k=1}^{\infty} B_k^{[2]} \right\|^{1/2} ;$$

$$(2) \left\| \sum_{k=1}^{\infty} (A_k + B_k)^{[2]} \right\|^{1/2} \leq \left\| \sum_{k=1}^{\infty} A_k^{[2]} \right\|^{1/2} + \left\| \sum_{k=1}^{\infty} B_k^{[2]} \right\|^{1/2} .$$

Now we have the Riesz-Fischer-type theorem.

**Theorem 3.6** *Both  $\mathcal{O}_b$  and  $\mathcal{O}_c$  equipped with the norm  $\|\cdot\|$  are Banach spaces.*

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Another space of sequences in  $\mathcal{S}^2$  which can reasonably be defined is the following:

$$\mathcal{O}_k := \left\{ \left\{ A_k = \begin{bmatrix} a_{ji}^{(k)} \end{bmatrix} \right\}_{k=1}^{\infty} \subset \mathcal{S}^2 : \text{the matrix } \begin{bmatrix} \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^2 \end{bmatrix} \text{ is compact} \right\}.$$

By the continuity of the map  $A \mapsto A^{[2]}$  from  $\mathcal{S}^2$  into  $\mathcal{B}(l_2)$ , we obtain that if  $\{A_k\}_{k=1}^{\infty} \in \mathcal{O}_k$ , then  $A_k^{[2]}$  is necessarily compact for all  $k$ .

**Theorem 3.7**  $\mathcal{O}_k \subsetneq \mathcal{O}_c$ .

**Theorem 3.8** *A sequence  $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$  belongs to  $\mathcal{O}_k$  if and only if  $A_k^{[2]}$  is compact for all  $k$  and  $\mathbf{A}$  belongs to  $\mathcal{O}_c$ .*

**Theorem 3.9** *The set  $\mathcal{O}_k$  is a Banach space under the norm  $\|\cdot\|_k$ .*

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## 4 Duality and Preduality of $\mathcal{O}_k$

**Theorem 4.1** *The space  $\mathcal{O}_k$  cannot be the dual space of any normed space.*

Let  $\mathcal{K}^2$  be the set of all infinite matrices  $A$  whose  $A^{[2]}$  is compact as an operator on  $l_2$ . For any matrix  $A = [a_{ji}]$ , let

$\sum A = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ji}$  if the sum  $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ji}$  converges. Let

$$\mathcal{M} = \left\{ B : \sum |A \bullet B| < \infty \text{ for all } A \in \mathcal{K}^2 \right\}.$$

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Let

$$\Delta = \left\{ \{B_k\}_{k=1}^{\infty} \subset \mathcal{M} : \sum_{k=1}^{\infty} \left| \sum A_k \bullet B_k \right| < \infty \text{ for all } \{A_k\}_{k=1}^{\infty} \in \mathcal{O}_k \right\}.$$

It can be shown by the closed graph theorem that

$$\|\{B_k\}_{k=1}^{\infty}\|_* := \sup \left\{ \sum_{k=1}^{\infty} \left| \sum A_k \bullet B_k \right| : \{A_k\}_{k=1}^{\infty} \in \mathcal{O}_k, \|\{A_k\}_{k=1}^{\infty}\| \leq 1 \right\}$$

is finite for all  $\{B_k\}_{k=1}^{\infty} \in \Delta$ .

**Theorem 4.2** *The set  $\Delta$  equipped with the norm  $\|\cdot\|_*$  is a Banach space.*

**Theorem 4.3**  *$\mathcal{O}_k^*$  is isometrically isomorphic to  $\Delta$ .*