A horizontal categorification of Gel'fand duality

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Joint Work

This is an ongoing joint research work with

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Abstract

In the setting of C*-categories, we provide a definition of "spectrum"of a commutative full C*-category as a one-dimensional unital saturated Fell bundle over a suitable groupoid (equivalence relation) and prove a categorical Gelfand duality theorem generalizing the usual Gelfand duality between the categories of commutative unital C*-algebras and compact Hausdorff spaces. Although many of the individual ingredients that appear along the way are well-known, the somehow unconventional way we "glue" them together seems to shed some new light on the subject.

Dualities

A **duality** (a contravariant equivalence) of two categories $\mathscr{C}$ and $\mathscr{D}$ is a pair of contravariant functors $\Gamma : \mathscr{C} \to \mathscr{D}$ and $\Sigma : \mathscr{D} \to \mathscr{C}$ such that $\Gamma \circ \Sigma$ and $\Sigma \circ \Gamma$ are naturally equivalent to the respective identity functors $\mathcal{I}_{\mathscr{D}}$ and $\mathcal{I}_{\mathscr{C}}$. A duality is actually specified by two functors, but given any one of the two functors in the dual pair, the other one is unique up two natural isomorphism. A functor Γ is in a duality pair if and only if it is full, faithful and representative.

Gel'fand Theorem 1

There exists a duality (Γ, Σ) between the category $\mathscr{T}^{(1)}$, of continuous maps between compact Hausdorff topological spaces, and the category $\mathscr{A}^{(1)}$, of unital homomorphisms of commutative unital C*-algebras, where

Gel'fand Theorem 1

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- Γ is the functor that to every compact Hausdorff topological space $X \in \mathrm{Ob}_{\mathscr{T}^{(1)}}$ associates the unital commutative C*-algebra $\mathrm{C}(X)$ of complex valued continuous functions on X (with pointwise multiplication and conjugation and supremum norm) and that to every continuous map $f : X \to Y$ associates the unital $*$-homomorphism $f^\bullet : C(Y) \to C(X)$ given by the pull-back of continuous functions by f;

Gel'fand Theorem 2

- Σ is the functor that to every unital commutative C*-algebra $\mathcal{A}$ associates its spectrum

  $$\mathrm{Sp}(\mathcal{A}) := \{\omega \mid \omega : \mathcal{A} \to \mathbb{C}, \text{ is a unital } *\text{-homomorphism}\}$$

  (as a topological space with the weak topology induced by the evaluation maps $\omega \mapsto \omega(x)$, for all $x \in \mathcal{A}$) and that to every unital $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ of algebras associates the continuous map $\phi^\bullet : \mathrm{Sp}(\mathcal{B}) \to \mathrm{Sp}(\mathcal{A})$ given by the pull-back under ϕ.

Gel'fand Theorem 3

- The natural isomorphism $\mathfrak{G} : \mathcal{I}_{\mathscr{A}(1)} \to \Gamma \circ \Sigma$ is given by the **Gel'fand transforms** $\mathfrak{G}_{\mathcal{A}} : \mathcal{A} \to C(\mathrm{Sp}(\mathcal{A}))$ defined by $\mathfrak{G}_{\mathcal{A}} : a \mapsto \hat{a}$ where $\hat{a} : \mathrm{Sp}(\mathcal{A}) \to \mathbb{C}$ is the Gelf'and transform of a i.e. $\hat{a} : \omega \mapsto \omega(a)$.

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- Similarly the natural isomorphism $\mathfrak{E} : \mathcal{I}_{\mathscr{T}(1)} \to \Sigma \circ \Gamma$ is given by the **evaluation** homeomorphisms $\mathfrak{E}_X : X \to \mathrm{Sp}(C(X))$ defined by $\mathfrak{E}_X : p \mapsto \mathrm{ev}_p$, where $\mathrm{ev}_p : C(X) \to \mathbb{C}$ is the p-evaluation i.e. $\mathrm{ev}_p : f \mapsto f(p)$.

Horizontal Categorification of Gel’fand Duality 1

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- suitable horizontal categorifications $\mathscr{T}$ of $\mathscr{T}^{(1)}$ and $\mathscr{A}$ of $\mathscr{A}^{(1)}$;
- to extend the categorical duality $(\Gamma^{(1)}, \Sigma^{(1)})$ between $\mathscr{T}^{(1)}$ and $\mathscr{A}^{(1)}$ of Gel'fand Theorem, to a natural categorical equivalence between $\mathscr{T}$ and $\mathscr{A}$:

$$\begin{tikzcd}
\mathscr{T}^{(1)} \arrow[r, harpoon, shift left, "\Gamma^{(1)}"] \arrow[d, hook] & \mathscr{A}^{(1)} \arrow[l, harpoon, shift left, "\Sigma^{(1)}"] \arrow[d, hook] \\
\mathscr{T} \arrow[r, harpoon, shift left, "\Gamma"] & \mathscr{A}. \arrow[l, harpoon, shift left, "\Sigma"]
\end{tikzcd}$$

Horizontal Categorification of Gel'fand Duality 2

In the setting of C*-categories, we provide a definition of the "spectrum" of a commutative full C*-category as a one-dimensional saturated unital Fell-bundle over a suitable groupoid (equivalence relation) and we prove a categorical Gel'fand duality theorem generalizing the usual Gel'fand duality between the categories of commutative C*-algebras and compact Hausdorff spaces.

C*-categories 1

A **C*-category**[1],[2] is a category $\mathcal{C}$ such that:

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- the compositions are bilinear maps,
- the norm satisfies:

$$\|xy\| \leq \|x\| \cdot \|y\|, \quad \forall x \in \mathcal{C}_{AB}, \ \forall y \in \mathcal{C}_{BC},$$
$$\|\iota_A\| = 1, \quad \forall A \in \mathrm{Ob}_{\mathcal{C}},$$

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C*-categories 2

- there is an involution $*:\mathrm{Hom}_{\mathcal{C}}\to\mathrm{Hom}_{\mathcal{C}}$ such that:

$$\begin{aligned}
&x^*\in\mathrm{Hom}_{\mathcal{C}}(B,A)\quad\forall x\in\mathrm{Hom}_{\mathcal{C}}(A,B),\\
&(\alpha x+\beta y)^*=\overline{\alpha}x^*+\overline{\beta}y^*\quad\forall\alpha,\beta\in\mathbb{C}\ \forall x,y\in\mathcal{C}_{AB},\\
&(xy)^*=y^*x^*\quad\forall y\in\mathcal{C}_{BC}\ \forall x\in\mathcal{C}_{AB},\\
&(x^*)^*=x\quad\forall x\in\mathcal{C}_{AB},\\
&\|x^*x\|=\|x\|^2\quad\forall x\in\mathcal{C}_{BA},\\
&x^*x\in\mathcal{C}_{AA+}\quad\forall x\in\mathcal{C}_{BA}.
\end{aligned}$$

C*-categories 3

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A standard example is the C*-category of bounded linear operators between Hilbert spaces.

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- a continuous norm $\|\cdot\|\colon E \to \mathbb{R}$, making all the fibers $E_x := p^{-1}(x)$ Banach spaces, and
- for all $x \in X$, a base of neighborhoods of $0_x \in E_x$ in the topology of E of the form

  $$B_{U,\varepsilon} := \{e \in E \mid p(e) \in U, \|e\| < \varepsilon\},$$

  where $\varepsilon > 0$ and U is a neighborhood of $x \in X$.

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If X is an involutive category, i.e. there is a map $* : X \to X$ with the properties $(x^*)^* = x$ and $(x \circ y)^* = y^* \circ x^*$, for all $(x, y) \in X^2$, we also require $*$ to be continuous.

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By a **topological involutive category** we mean an involutive category with a topology such that the composition and the involution continuous.

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It is easy to see that for every $x = p(e^*e)$, E_x is a C*-algebra.

Fell Bundles 3

Note that the fiber E_x has a natural structure of Hilbert C*-bimodule over the C*-algebras $E_{r(x)}$ on the left and $E_{s(x)}$ on the right, where $r(x)$ and $s(x)$ denote the range and the source of the morphisms x in the category X.

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A Fell bundle is said to be **saturated** if the above Hilbert C*-bimodules E_x are full.

Note also that in a saturated Fell bundle, the Hilbert C*-bimodules E_x are imprimitivity bimodules.

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Definition

A **topological spaceoid** $(\mathcal{E}, \pi, \mathcal{X})$ is a unital rank-one Fell bundle over the product involutive inverse topological category $\mathcal{X} := \Delta_X \times \mathcal{R}_{\mathcal{O}}$.

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- $f := (f_\Delta, f_\mathcal{R})$ with $f_\Delta : \Delta_1 \to \Delta_2$ a continuous map of topological spaces and $f_\mathcal{R} : \mathcal{R}_1 \to \mathcal{R}_2$ an isomorphism of equivalence relations;

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[a] Where $\mathcal{X}_j = \Delta_{X_j} \times \mathcal{R}_{\mathcal{O}_j}$, with $\mathcal{O}_j$ sets and X_j compact Hausdorff spaces for $j = 1, 2$.

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- $\mathcal{F} : f^\bullet(\mathcal{E}_2) \to \mathcal{E}_1$ is a fiberwise linear $*$-functor such that $\pi_1 \circ \mathfrak{F} = (\pi_2)^f$, where $(f^\bullet(\mathcal{E}_2), \pi_2^f, \mathcal{X}_1)$ denotes the standard f-pull-back of $(\mathcal{E}_2, \pi_2, \mathcal{X}_2)$.

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Category of Spaceoids 1

Topological spaceoids constitute a category if composition is defined by

$$(g, \mathcal{G}) \circ (f, \mathcal{F}) := (g \circ f, \mathcal{F} \circ f^\bullet(\mathcal{G}))$$

with identities given by

$$\iota_{(\mathcal{E},\pi,\mathcal{X})} := (\iota_\mathcal{X}, \iota_\mathcal{E}).$$

Note here that, with abuse of notation, $f^\bullet(g^\bullet(\mathcal{E}_3))$ is naturally identified with the standard $(g \circ f)$-pull-back of $(\mathcal{E}_3, \pi_3, \mathcal{X}_3)$ and that $(\mathcal{E}, \pi, \mathcal{X})$ is identified with the standard $\iota_\mathcal{X}$-pull-back of itself.

Category of Spaceoids 2

The category $\mathscr{T}^{(1)}$ of continuous maps between compact Hausdorff spaces can be naturally identified with the full subcategory of the category $\mathscr{T}$ of spaceoids with index set $\mathcal{O}$ containing a single element.

- To every object $X \in \mathrm{Ob}_{\mathscr{T}^{(1)}}$ we associate the trivial $\mathbb{C}$-line bundle $\mathcal{X}_X \times \mathbb{C}$ over the involutive category $\mathcal{X}_X := \Delta_X \times \mathcal{R}_{\mathcal{O}_X}$ with $\mathcal{O}_X := \{X\}$ the one point set.
- To every continuous map $f : X \to Y$ in $\mathscr{T}^{(1)}$ we associate the morphism $(g, \mathcal{G})$ with $g_\Delta(p,p) := (f(p), f(p))$, $g_\mathcal{R} : (X,X) \mapsto (Y,Y)$ and $\mathcal{G} := \iota_{\mathcal{X}_X \times \mathbb{C}}$.

Note that the trivial bundle over $\mathcal{X}_X$ is naturally identified with the standard f-pull-back of the trivial bundle over $\mathcal{X}_Y$; hence $\mathcal{G}$ can be taken as the identity map.

$$\begin{tikzcd}\mathscr{T} \arrow[r, shift left, "\Gamma"] & \color{red}{\mathscr{A}} \arrow[l, shift left, "\Sigma"]\end{tikzcd}$$

The Category of Small C*-categories

Let $\mathscr{A}$ be the category whose objects consist of full commutative small C*-categories.

For any two full commutative small C*-categories $\mathcal{C}$ and $\mathcal{D}$ (with the same cardinality of the set of objects), a morphism $\Phi : \mathcal{C} \to \mathcal{D}$ is an object bijective $*$-functor i.e. a map such that

$$\begin{aligned}
&\Phi(\alpha x + \beta y) = \alpha\Phi(x) + \beta\Phi(y) \quad \forall x, y \in \mathcal{C}_{AB}\, \forall \alpha, \beta \in \mathbb{C},\\
&\Phi(x \circ y) = \Phi(x) \circ \Phi(y) \quad \forall x \in \mathcal{C}_{CB}\, \forall y \in \mathcal{C}_{BA},\\
&\Phi(x^*) = \Phi(x)^* \quad \forall x \in \mathcal{C}_{AB},\\
&\Phi(\iota) \in \mathcal{D}_o \quad \forall \iota \in \mathcal{C}_o,\\
&\Phi_o := \Phi|_{\mathcal{C}_o} : \mathcal{C}_o \to \mathcal{D}_o \quad \text{is bijective},
\end{aligned}$$

where $\mathcal{C}_o$ and $\mathcal{D}_o$ are the sets of identities of $\mathcal{C}$ and $\mathcal{D}$, respectively.

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The Section Functor Γ on Objects

To every spaceoid $(\mathcal{E}, \pi, \mathcal{X})$, with $\mathcal{X} := \Delta_X \times \mathcal{R}_{\mathcal{O}}$, we can associate a full commutative C*-category $\Gamma(\mathcal{E})$ as follows:

- $\mathrm{Ob}_{\Gamma(\mathcal{E})} := \mathcal{O}$;
- $\forall A, B \in \mathrm{Ob}_{\Gamma(\mathcal{E})}, \quad \mathrm{Hom}_{\Gamma(\mathcal{E})}(B, A) := \Gamma(\Delta_X \times \{(A, B)\}; \mathcal{E})$, where $\Gamma(\Delta_X \times \{(A, B)\}; \mathcal{E})$ denotes the set of continuous sections $\sigma : \Delta_X \times \{(A, B)\} \to \mathcal{E}$, $\sigma : p_{AB} \mapsto \sigma_p^{AB} \in \mathcal{E}_{p_{AB}}$ of the restriction of $\mathcal{E}$ to the base space $\Delta_X \times \{(A, B)\} \subset \mathcal{X}$.
- for all $\sigma \in \mathrm{Hom}_{\Gamma(\mathcal{E})}(C, B)$ and $\rho \in \mathrm{Hom}_{\Gamma(\mathcal{E})}(B, A)$:

$$\rho \circ \sigma : p_{AC} \mapsto (\rho \circ \sigma)_p^{AC} := \rho_p^{AB} \circ \sigma_p^{BC},$$
$$\rho^* : p_{BA} \mapsto (\rho^*)_p^{BA} := (\rho_p^{AB})^*,$$
$$\|\rho\| := \sup_{p \in \Delta_X} \|\rho_p^{AB}\|_{\mathcal{E}},$$

with operations taken in the total space $\mathcal{E}$ of the Fell bundle.

The Section Functor Γ on Morphisms

We extend now the definition of Γ to the morphism of $\mathscr{T}$ in order to obtain a contravariant functor.
Let $(f,\mathcal{F})$ be a morphism in $\mathscr{T}$ from $(\mathcal{E}_1,\pi_1,\mathcal{X}_1)$ to $(\mathcal{E}_2,\pi_2,\mathcal{X}_2)$.
Given $\sigma\in\Gamma(\mathcal{E}_2)$, we consider the unique section $f^\bullet(\sigma):\mathcal{X}_1\to f^\bullet(\mathcal{E}_2)$ such that $f^{\pi_2}\circ f^\bullet(\sigma)=\sigma\circ f$ and the composition $\mathcal{F}\circ f^\bullet(\sigma)$.
In this way we get a map

$$\Gamma_{(f,\mathcal{F})}:\Gamma(\mathcal{E}_2)\to\Gamma(\mathcal{E}_1),\quad \Gamma_{(f,\mathcal{F})}:\sigma\mapsto\mathcal{F}\circ f^\bullet(\sigma),\quad \forall\sigma\in\Gamma(\mathcal{E}_2).$$

Proposition

For any morphism $(\mathcal{E}_1,\pi_1,\mathcal{X}_1)\xrightarrow{(f,\mathcal{F})}(\mathcal{E}_2,\pi_2,\mathcal{X}_2)$ *in* $\mathscr{T}$, *the map* $\Gamma_{(f,\mathcal{F})}:\Gamma(\mathcal{E}_2)\to\Gamma(\mathcal{E}_1)$ *is a morphism in* $\mathscr{A}$.
The pair of maps $\Gamma:(\mathcal{E},\pi,\mathcal{X})\mapsto\Gamma(\mathcal{E})$ *and* $\Gamma:(f,\mathcal{F})\mapsto\Gamma_{(f,\mathcal{F})}$ *gives a contravariant functor from the category* $\mathscr{T}$ *of spaceoids to the category* $\mathscr{A}$ *of small full commutative* C^**-categories.*

$$\begin{tikzcd}
\mathscr{T} \arrow[r, shift left, "\Gamma"] & \mathscr{A} \arrow[l, shift left, "\Sigma"]
\end{tikzcd}$$

The Spectrum functor Σ on Objects 1

We proceed to associate to every commutative full C*-category $\mathcal{C}$ its spectral spaceoid $\Sigma(\mathcal{C}) := (\mathcal{E}^{\mathcal{C}}, \pi^{\mathcal{C}}, \mathcal{X}^{\mathcal{C}})$.

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- By definition two $*$-functors $\omega_1, \omega_2 \in [\mathcal{C}; \mathbb{C}]$ are **unitarily equivalent** if there exists a "unitary" natural trasformation $A \mapsto \nu_A \in \mathbb{T}$ between them. This is true iff $\omega_1|_{\mathcal{C}_{AA}} = \omega_2|_{\mathcal{C}_{AA}}$ for all $A \in \mathrm{Ob}_{\mathcal{C}}$.

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- Let $\mathrm{Sp}_b(\mathcal{C}) := \{[\omega] \mid \omega \in [\mathcal{C}; \mathbb{C}]\}$ denote the **base spectrum** of $\mathcal{C}$, defined as the set of unitary equivalence classes of $*$-functors in $[\mathcal{C}; \mathbb{C}]$. It is a compact Hausdorff space with the quotient topology induced by the map $\omega \mapsto [\omega]$.

The Spectrum functor Σ on Objects 2

- Let $\mathcal{X}^{\mathcal{C}} := \Delta^{\mathcal{C}} \times \mathcal{R}^{\mathcal{C}}$ be the direct product of the compact Hausdorff $*$-category $\Delta^{\mathcal{C}} := \Delta_{\mathrm{Sp}_b(\mathcal{C})}$ and the topologically discrete $*$-category $\mathcal{R}^{\mathcal{C}} := \mathcal{C}/\mathcal{C} \simeq \mathcal{R}_{\mathrm{Ob}_{\mathcal{C}}}$.
- For $\omega \in [\mathcal{C}; \mathbb{C}]$, the set $\mathcal{J}_\omega := \{x \in \mathcal{C} \mid \omega(x) = 0\}$ is an ideal in $\mathcal{C}$ and $\mathcal{J}_{\omega_1} = \mathcal{J}_{\omega_2}$ if $[\omega_1] = [\omega_2]$.
- Denoting $[\omega]_{AB}$ the point $([\omega], (A, B)) \in \mathcal{X}^{\mathcal{C}}$, we define:

$$\mathcal{J}_{[\omega]_{AB}} := \mathcal{J}_\omega \cap \mathcal{C}_{AB}, \quad \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}} := \frac{\mathcal{C}_{AB}}{\mathcal{J}_{[\omega]_{AB}}}, \quad \mathcal{E}^{\mathcal{C}} := \biguplus_{[\omega]_{AB} \in \mathcal{X}^{\mathcal{C}}} \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}}.$$

Define the map $\pi^{\mathcal{C}} : \mathcal{E}^{\mathcal{C}} \to \mathcal{X}^{\mathcal{C}}$ by sending $e \in \mathcal{E}^{\mathcal{C}}_{[\omega]_{AB}}$ to $[\omega]_{AB} \in \mathcal{X}^{\mathcal{C}}$.

Proposition

The triple $(\mathcal{E}^{\mathcal{C}}, \pi^{\mathcal{C}}, \mathcal{X}^{\mathcal{C}})$ is a spaceoid.

The Spectrum functor Σ on Morphisms 1.

If $\Phi : \mathcal{C} \to \mathcal{D}$ is an object-bijective $*$-functor between two small commutative full C*-categories with spaceoids $\Sigma(\mathcal{C}), \Sigma(\mathcal{D}) \in \mathscr{T}$, we can define a morphism $\Sigma^\Phi : \Sigma(\mathcal{D}) \to \Sigma(\mathcal{C})$ in the category $\mathscr{T}$ and obtain the following result.

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Proposition

For any morphism $\mathcal{C} \xrightarrow{\Phi} \mathcal{D}$ *in* $\mathscr{A}$, *the map* $\Sigma(\mathcal{D}) \xrightarrow{\Sigma^\Phi} \Sigma(\mathcal{C})$ *is a morphism of spectral spaceoids. The pair of maps* $\Sigma : \mathcal{C} \mapsto \Sigma(\mathcal{C})$ *and* $\Sigma : \Phi \mapsto \Sigma^\Phi$ *give a contravariant functor* $\Sigma : \mathscr{A} \to \mathscr{T}$, *from the category* $\mathscr{A}$ *of object-bijective* $*$*-functors between small commutative full C*-categories to the category* $\mathscr{T}$ *of spaceoids.*

Gel'fand Duality Theorem for C*-categories.

Theorem (Bertozzini-Conti-L.)

There exists a duality (Γ, Σ) *between the category* $\mathscr{T}$ *of object-bijective morphisms between spaceoids and the category* $\mathscr{A}$ *of object-bijective* $*$*-functors between small commutative full* C^**-categories, where*

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- Γ *is the functor that to every spaceoid* $(\mathcal{E}, \pi, \mathcal{X}) \in \mathrm{Ob}_{\mathscr{T}}$ *associates the small commutative full* C^**-category* $\Gamma(\mathcal{E})$ *and that to every morphism between spaceoids* $(f, \mathcal{F}) : (\mathcal{E}_1, \pi_1, \mathcal{X}_1) \to (\mathcal{E}_2, \pi_2, \mathcal{X}_2)$ *associates the* $*$*-functor* $\Gamma_{(f,\mathcal{F})}$*;*

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- Σ *is the functor that to every small commutative full* C^**-category* $\mathcal{C}$ *associates its spectral spaceoid* $\Sigma(\mathcal{C})$ *and that to every object-bijective* $*$*-functor* $\Phi : \mathcal{C} \to \mathcal{D}$ *of* C^**-categories in* $\mathscr{A}$ *associates the morphism* $\Sigma^{\Phi} : \Sigma(\mathcal{D}) \to \Sigma(\mathcal{C})$ *between spaceoids.*

Generalizations and Applications of Gel'fand Duality 1.

We are now working on a number of generalizations of our horizontal categorified Gel'fand duality:

- Gel'fand duality for general $*$-functors and $*$-relators.
- Gel'fand duality for non-full C*-categories.
- Pontryagin duality for commutative groupoids.
- Continuous functional calculus and a spectral theorem for bounded linear operators between Hilbert spaces.
- Categorification of Dauns-Hofmann spectral theorem and dualities for non-commutative C*-categories or more generally higher rank Fell bundles.
- Gel'fand dualities for commutative higher C*-categories and "higher-spaceoids".[3]
- Spectral triples over C*-categories and horizontal categorification of spectral triples and other spectral geometries.

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[3] Very interesting is the possible relation between such "higher" spectra and the notions of stacks and gerbes already used in higher gauge theory.

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http://arxiv.org/abs/0812.3601

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