INTRINSIC 3d-ELASTICITY

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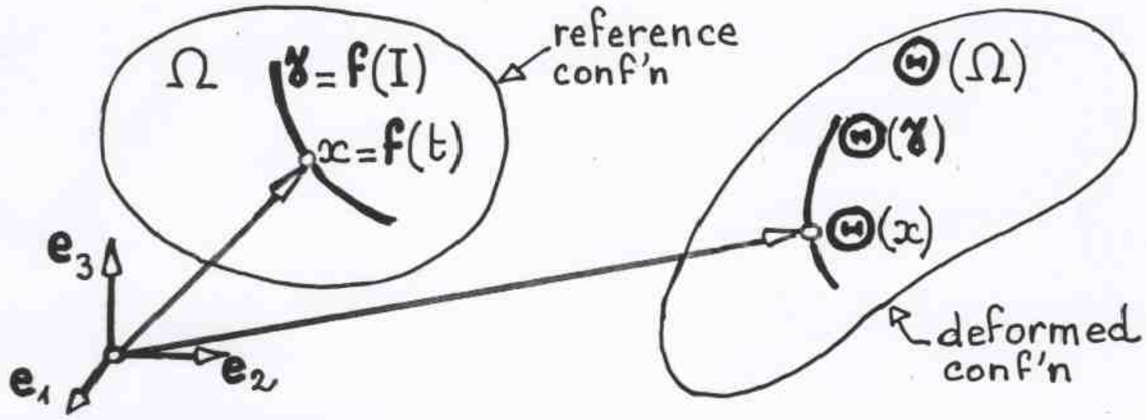
1. THE METRIC, OR CAUCHY-GREEN, TENSOR

$\mathbb{M}^3 = \{3\times 3 \text{ real matrices}\}$ $\quad \mathbb{S}^3_> = \{\mathbf{A} \in \mathbb{M}^3 ; \mathbf{A} = \mathbf{A}^T \text{ pos-def}\}$

$\mathbb{O}^3 = \{\mathbf{Q} \in \mathbb{M}^3 ; \mathbf{Q}\mathbf{Q}^T = \mathbf{I}\}$ $\quad \mathbb{O}^3_+ = \{\mathbf{Q} \in \mathbb{O}^3 ; \det \mathbf{Q} = 1\}$

- **GEOMETRY** Ω: open subset $\subset \mathbb{R}^3$



$\boxed{\boldsymbol{\Theta} : \Omega \to \mathbb{R}^3}$ smooth enough immersion: "deformation"

$$\boxed{\mathbf{C} = \nabla\boldsymbol{\Theta}^T \nabla\boldsymbol{\Theta} = (g_{ij}) : \Omega \to \mathbb{S}^3_>}$$

"metric tensor"

"Cauchy-Green tensor" $\qquad g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta}$

$\boldsymbol{\gamma} = \mathbf{f}(I) \subset \Omega$, $I \subset \mathbb{R}$, $\mathbf{f}(t) = f^i(t)\mathbf{e}_i$, $t \in I$

$$\text{Length } \boldsymbol{\Theta}(\boldsymbol{\gamma}) = \int_I \sqrt{g_{ij}(\mathbf{f}(t)) \frac{df^i}{dt}(t) \frac{df^j}{dt}(t)}\, dt$$

2. Classical and "intrinsic" approaches in 3D-Nonlinear Elasticity

CLASSICAL

$$\boxed{I(\Theta) = \int_\Omega W(x, \overbrace{\nabla\Theta(x)^T \nabla\Theta(x)}^{\mathbf{C}(x)})\,dx - \int_\Omega \mathbf{f}(x)\cdot\Theta(x)\,dx}$$

energy — dependence on $\nabla\Theta$ via $\mathbf{C} = \nabla\Theta^T\nabla\Theta$ (frame-indifference) — applied force

$$\boxed{\inf I(\Theta) \text{ on } \begin{cases} \Theta \text{ injective in } \Omega \text{ (non-interpenetrability)} \\ \det\nabla\Theta > 0 \text{ in } \Omega \text{ (orientation-preserving)} \\ \Theta = \Theta_0 \text{ on } \Gamma_0 \subset \partial\Omega \text{ (boundary condition)} \end{cases}}$$

Existence via "polyconvexity" J. Ball, ARMA (1977)

INTRINSIC

Question: Why not $\mathbf{C}: \Omega \to \mathbb{S}^3$ as the primary unknown instead of $\Theta: \Omega \to \mathbb{R}^3$? S.S. Antman, ARMA (1976)

But then: Quid $\int_\Omega \mathbf{f}(x)\cdot\Theta(x)\,dx$? Quid $\Theta = \Theta_0$ on Γ_0?

Minimization problem with constraints on $\mathbf{C}$ $\left(R^p_{\cdot ijk} = 0\right)$

> Pbm#1 : Given $\mathbf{C}: \Omega \to \mathbb{S}^3_>$: recovery of $\Theta: \Omega \to \mathbb{R}^3$ such that $\nabla\Theta^T\nabla\Theta = \mathbf{C}$ in Ω?
>
> Pbm#2 : Continuity of $\mathbf{C} \to \Theta$ ∴ which topologies?
> Minimizing seq'ce: $\mathbf{C}^k \to \mathbf{C} \Rightarrow \Theta^k \to \Theta$?

Appl'ns: Differential geometry, 3d-nonlinear elasticity

Same questions for surfaces in $\mathbb{R}^3$ as functions of their two fundamental forms

Appl'ns: Differential geometry, nonlinear shell theory

Note : Rotation field $\mathbf{R}$ in $\nabla\Theta = \overbrace{\mathbf{R}\,\mathbf{C}^{1/2}}^{\text{polar factorization}}$ may be also chosen as a primary unknown (see p. $\boxed{11 \text{ bis}}$)

P.G.C. - L. Gratie - O. Iosifescu - C. Mardare - C. Vallée, JMPA (2006)

3. Recovery of $\Theta : \Omega \to \mathbb{R}^n$ from $\mathbf{C} : \Omega \to \mathbb{S}^n$

Existence Theorems: Let $\mathbf{C} = (g_{ij})$ symmetric & positive-definite in Ω

$$\Gamma_{ij}^{k} \overset{\text{def}}{=} \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}) = \Gamma_{ji}^{k}, \quad (g^{k\ell}) \overset{\text{def}}{=} (g_{ij})^{-1}$$

$$\boxed{R^{p}_{\cdot ijk} \overset{\text{def}}{=} \partial_j \Gamma_{ik}^{p} - \partial_k \Gamma_{ij}^{p} + \Gamma_{ik}^{\ell} \Gamma_{j\ell}^{p} - \Gamma_{ij}^{\ell} \Gamma_{k\ell}^{p} = 0 \text{ in } \Omega}$$ nec. cond'n if $\mathbf{C} = \nabla\Theta^T \nabla\Theta$

Thm $\Omega \subset \mathbb{R}^n$: simply-connected, open

Existence of $\Theta : \Omega \to \mathbb{R}^n$ s.t. $\nabla\Theta^T \nabla\Theta = \mathbf{C}$ in Ω:

(1) $\boxed{\mathbf{C} \in \mathcal{C}^2(\Omega) \to \Theta \in \mathcal{C}^3(\Omega)}$ if $\boxed{R^{p}_{\cdot ijk} = 0 \text{ in } \Omega}$ (classical)

(2) $\boxed{\mathbf{C} \in \mathcal{C}^1(\Omega) \to \Theta \in \mathcal{C}^2(\Omega)}$ if $\boxed{R^{p}_{\cdot ijk} = 0 \text{ in } \mathcal{D}'(\Omega)}$
C. Mardare, Anal. Appl. (2003)

(3) $\boxed{\mathbf{C} \in \mathbf{W}^{1,\infty}_{loc}(\Omega) \to \Theta \in \mathbf{W}^{2,\infty}_{loc}(\Omega)}$ if $R^{p}_{\cdot ijk} = 0$ in $\mathcal{D}'(\Omega)$ &
$(g_{ij})^{-1} \in \mathbf{L}^\infty(\Omega)$ S. Mardare, Anal. Appl. (2004)

(4) $\boxed{\mathbf{C} \in \mathcal{C}^2(\overline{\Omega}) \to \Theta \in \mathcal{C}^3(\overline{\Omega})}$ if $R^{p}_{\cdot ijk} = 0$ in Ω &
$\mathbf{C}$ positive-definite in $\overline{\Omega}$ & $\partial\Omega$ Lipschitz-cont's
P.G. Ciarlet - C. Mardare, J. Math. Pures Appl. (2004)

(5) $\boxed{\mathbf{C} \in \mathbf{W}^{1,p}_{loc}(\Omega) \to \Theta \in \mathbf{W}^{2,p}_{loc}(\Omega) \text{ for any } p > n}$ (optimal)
S. Mardare, Advances in Differential Eqs. (2007)

Thm. $\Omega \subset \mathbb{R}^n$: simply-connected, bounded, $\underbrace{\text{smooth bdry}}$ manifold of class $\mathcal{C}^\infty$

$$\mathbf{C} \in \overbrace{\left\{ \mathbf{C} \in \mathbf{W}^{m,p}(\Omega);\ \mathbf{C}(x) \in \mathbb{S}^n_> \text{ for all } x \in \overline{\Omega},\ R^{p}_{\cdot ijk} = 0 \text{ in } \mathcal{D}'(\Omega) \right\} \longrightarrow \Theta \in \mathbf{W}^{m+1,p}(\overline{\Omega})}$$

is of class $\mathcal{C}^\infty$ for any $m > 1$, $p > 1$, s.t. $p(m-1) > n$.

C. Mardare, Anal. Appl. (2006)

4. Recovery of $\boldsymbol{\Theta}:\Omega \to \mathbb{R}^n$ from $\mathbf{C}:\Omega \to \mathbb{S}^n$: Uniqueness Theorems, a.k.a. "Rigidity Theorems":

Thm Ω: connected open subset of $\mathbb{R}^n$ (1) $\boldsymbol{\Theta}$ immersion
$\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega)$ & $\widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^1(\Omega)$ s.t. $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ in Ω. Then:

$\boxed{\exists \mathbf{c} \in \mathbb{R}^n, \exists \mathbf{Q} \in \mathbb{O}^n \text{ s.t. } \widetilde{\boldsymbol{\Theta}}(x) = \mathbf{c} + \mathbf{Q}\boldsymbol{\Theta}(x), x \in \Omega}$ (classical)

(2) $\left.\begin{array}{l} \boldsymbol{\Theta} \in \mathcal{C}^1(\Omega) \,\&\, \det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0 \text{ in } \Omega \\ \widetilde{\boldsymbol{\Theta}} \in \mathbf{H}^1(\Omega) \,\&\, \det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0 \text{ a.e. in } \Omega \end{array}\right\}$ s.t. $\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta}$ a.e. in Ω. Then:

$\boxed{\exists \mathbf{c} \in \mathbb{R}^n, \exists \mathbf{Q} \in \mathbb{O}^n_+ \text{ s.t. } \widetilde{\boldsymbol{\Theta}}(x) = \mathbf{c} + \mathbf{Q}\boldsymbol{\Theta}(x) \text{ for a.a. } x \in \Omega}$

P.G. Ciarlet – C. Mardare, Math. Models Methods Appl. Sci. (2003)

Rk. Information on sign of $\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}$ is essential in (2)!

Pbm: Given $\boldsymbol{\Theta} \in \mathbf{H}^1(\Omega)$ s.t. $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ a.e. in Ω, identify the set:

$$\left\{\widetilde{\boldsymbol{\Theta}} \in \mathbf{H}^1(\Omega);\ \det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0 \text{ a.e. in } \Omega, \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^T \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T \boldsymbol{\nabla}\boldsymbol{\Theta} \text{ a.e. in } \Omega\right\},$$

which is **strictly larger** than the set

$$\left\{\widetilde{\boldsymbol{\Theta}} \in \mathbf{H}^1(\Omega); \exists \mathbf{c} \in \mathbb{R}^n, \exists \mathbf{Q} \in \mathbb{O}^n_+, \widetilde{\boldsymbol{\Theta}}(x) = \mathbf{c} + \mathbf{Q}\boldsymbol{\Theta}(x) \text{ for a.a. } x \in \Omega\right\}!$$

5. CONTINUITY OF $\mathbf{C} \to \Theta$

$|\mathbf{b}| = \sqrt{\mathbf{b}^T\mathbf{b}}$, $\mathbf{b} \in \mathbb{R}^n$ $\qquad |\mathbf{A}| = \sup_{|\mathbf{b}|=1} |\mathbf{A}\mathbf{b}|$, $\mathbf{A} \in \mathbb{M}^n$

Thm $\dot{\Theta}$ = equivalence class of Θ mod $\mathcal{R}$, where
$(\tilde{\Theta}, \Theta) \in \mathcal{R} \Leftrightarrow \exists\, \mathbf{c} \in \mathbb{R}^n, \exists\, \mathbf{Q} \in \mathbb{O}^n$ s.t. $\tilde{\Theta}(x) = \mathbf{c} + \mathbf{Q}\Theta(x)$, $x \in \Omega$

(1) $\boxed{\mathbf{C} \in \mathcal{C}^2(\Omega) \to \dot{\Theta} \in \mathcal{C}^3(\Omega)/\mathcal{R}}$ Continuity wrt Fréchet topologies* of $\mathcal{C}^\ell(\Omega)$: Let $\Theta^k \in \mathcal{C}^3(\Omega)$, $k \geq 1$, and $\Theta \in \mathcal{C}^3(\Omega)$ be immersions s.t. $(\nabla\Theta^k)^T\nabla\Theta^k \to \nabla\Theta^T\nabla\Theta$ in $\mathcal{C}^2(\Omega)$. Then
$$\exists\, \tilde{\Theta}^k = \mathbf{c}^k + \mathbf{Q}^k\Theta^k,\ \mathbf{c}^k \in \mathbb{R}^n, \mathbf{Q}^k \in \mathbb{O}^n,\quad \tilde{\Theta}^k \to \Theta \text{ in } \mathcal{C}^3(\Omega).$$

* $\psi^k \to \psi$ in $\mathcal{C}^\ell(\Omega) \Leftrightarrow \forall K \Subset \Omega$, $\sup_{\substack{x \in K \\ |\alpha| \leq \ell}} |\partial^\alpha(\psi^k - \psi)(x)| \to 0$.

P.G. Ciarlet - F. Laurent, ARMA (2003)

(2) $\boxed{\mathbf{C} \in \mathcal{C}^2(\bar{\Omega}) \to \dot{\Theta} \in \mathcal{C}^3(\bar{\Omega})/\mathcal{R}}$ Local Lipschitz-continuity wrt Banach space norms $\sup_{\substack{x \in \bar{\Omega} \\ |\alpha| \leq \ell}} |\partial^\alpha\psi(x)|$ if Ω is bdd & $\partial\Omega$ is Lipschitz-cont's

P.G. Ciarlet - C. Mardare, J. Math. Pures Appl. (2004)

(3) $\boxed{\mathbf{C} \in \mathbf{L}^{p/2}(\Omega) \to \dot{\Theta} \in \mathbf{W}^{1,p}(\Omega)/\mathcal{R}}$ for $p \geq 2$

See next theorem

Rk. Continuity $\Theta \in \mathbf{W}^{1,p}(\Omega) \to \mathbf{C} = \nabla\Theta^T\nabla\Theta \in \mathbf{L}^{p/2}(\Omega)$ is clear for any $p \geq 2$.

$$\tilde{\Theta} \in \dot{\Theta} \iff \exists\, \mathbf{c} \in \mathbb{R}^n, \exists\, \mathbf{Q} \in \mathbb{O}^n_+,\ \tilde{\Theta}(x) = \mathbf{c} + \mathbf{Q}\Theta(x), x \in \Omega$$

$$\|\mathbf{F}\|_{\mathbf{L}^p(\Omega;\mathbb{M}^n)} = \left\{\int_\Omega |\mathbf{F}(x)|^p dx\right\}^{1/p} \quad \begin{pmatrix}\text{Rotational invariance:} \\ \forall \mathbf{Q} \in \mathbb{O}^n,\ |\mathbf{Q}\psi| = |\psi|\end{pmatrix}$$

$$\|\Theta\|_{\mathbf{H}^1(\Omega;\mathbb{R}^n)} = \left\{\int_\Omega \left(|\Theta(x)|^2 + |\nabla\Theta(x)|^2\right) dx\right\}^{1/2}$$

Thm P.G. Ciarlet - C. Mardare, J. Nonlinear Science (2004)

$\Omega \subset \mathbb{R}^n$: bounded, connected, open with $\partial\Omega$ Lipschitz-cont's

Let $\Theta \in \mathcal{C}^1(\bar{\Omega};\mathbb{R}^n)$ be s.t. $\det\nabla\Theta > 0$ in $\bar{\Omega}$. Then $\exists$ constant $c(\Theta)$ s.t.

$\forall\, \phi \in \mathbf{H}^1(\Omega;\mathbb{R}^n)$ s.t. $\det\nabla\phi > 0$ a.e. in Ω,

$\exists\, \tilde{\phi} \in \dot{\phi}$ such that

(*) $$\|\tilde{\phi} - \Theta\|_{\mathbf{H}^1(\Omega;\mathbb{R}^n)} \le c(\Theta)\, \|\nabla\phi^T\nabla\phi - \nabla\Theta^T\nabla\Theta\|^{1/2}_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}$$

(*) is a NONLINEAR KORN'S INEQUALITY

Cor Let $\Theta \in \mathcal{C}^1(\bar{\Omega};\mathbb{R}^n)$ be s.t. $\det\nabla\Theta > 0$ in $\bar{\Omega}$.

Let $\Theta^k \in \mathbf{H}^1(\Omega;\mathbb{R}^n)$ be s.t. $\det\nabla\Theta^k > 0$ a.e. in Ω, $k \geq 1$. Then:

$$\left.\begin{array}{c}(\nabla\Theta^k)^T\nabla\Theta^k \to \nabla\Theta^T\nabla\Theta \\ \text{in } \mathbf{L}^1(\Omega;\mathbb{S}^n)\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\exists\, \tilde{\Theta}^k \in \dot{\Theta}^k \\ \text{s.t. } \tilde{\Theta}^k \to \Theta \text{ in } \mathbf{H}^1(\Omega;\mathbb{R}^n)\end{array}\right.$$

Same result holds with $\mathbf{H}^1(\Omega;\mathbb{R}^n)$ replaced by $\mathbf{W}^{1,p}(\Omega;\mathbb{R}^n)$

-d°- $\mathbf{L}^1(\Omega;\mathbb{R}^n)$ -d°- $\mathbf{L}^{p/2}(\Omega;\mathbb{R}^n)$

for any $p \geq 2$.

COMMENTS (1) Earlier results of F. John (1961, 1972) and R.V. Kohn (1982) when $\boldsymbol{\Theta} = \mathbf{id}$ & $\boldsymbol{\Phi}$ is bi-Lipschitz

(2) Yu. G. Reshetnyak, Siberian Math. J. (2003):
Let $\boldsymbol{\Theta}^k \in \mathbf{W}^{1,\infty}(\Omega; \mathbb{R}^n)$, $k \geq 1$, be such that:

$$\begin{cases} \exists\, L \text{ s.t. } L^{-1}|\boldsymbol{\xi}| \leq |\boldsymbol{\nabla\Theta}^k(x)\boldsymbol{\xi}| \leq L\,|\boldsymbol{\xi}| \text{ for all } k \geq 1,\ \boldsymbol{\xi} \in \mathbb{R}^n, \text{ a.a. } x \in \Omega\ ^* \\ \text{for almost all } x \in \Omega,\quad \mathbf{C}(x) \overset{\text{def}}{=} \lim_{k\to\infty} (\boldsymbol{\nabla\Theta}^k(x))^T \boldsymbol{\nabla\Theta}^k(x) \text{ exists.} \end{cases}$$

Then $\exists\, \boldsymbol{\Theta} \in \mathbf{W}^{1,\infty}(\Omega; \mathbb{R}^n)$ & $\exists\, \widetilde{\boldsymbol{\Theta}}^k = \mathbf{c}^k + \mathbf{Q}^k \boldsymbol{\Theta}^k$, $\mathbf{c}^k \in \mathbb{R}^n$, $\mathbf{Q}^k \in \mathbb{O}^n$ s.t.
$\boldsymbol{\nabla\Theta}^T \boldsymbol{\nabla\Theta} = \mathbf{C}$ a.e. in Ω & $\widetilde{\boldsymbol{\Theta}}^k \xrightarrow[k\to\infty]{} \boldsymbol{\Theta}$ in $\mathbf{W}^{1,p}(\Omega; \mathbb{R}^n)$ for any $p > 1$.

(3) Some additional assumptions required in all cases:
bi-Lipschitz or *quasi-isometric mappings; smooth limit

6. Proof of the nonlinear Korn inequality (*)

Lemma 1: "Gradients suffice" Ass'ns on Ω & Θ as in Thm

Ineq. holds $\Longleftrightarrow$ $\exists$ constant $C(\Theta)$ s.t.

$\forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega;\mathbb{R}^n)$ s.t. $\det \nabla\boldsymbol{\phi} > 0$ a.e. in Ω, $\exists\, \mathbf{R} = \mathbf{R}(\boldsymbol{\phi},\Theta) \in \mathbb{O}^n_+$ s.t.

$$(**)\quad \|\nabla\boldsymbol{\phi} - \mathbf{R}\nabla\Theta\|_{\mathbf{L}^2(\Omega;\mathbb{M}^n)} \leq C(\Theta)\,\|\nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi} - \nabla\Theta^T\nabla\Theta\|^{1/2}_{\mathbf{L}^1(\Omega;\mathbb{S}^n)}$$

Pf. Generalized Poincaré inequality: $\exists$ constant d s.t.:

$$\forall \boldsymbol{\psi} \in \mathbf{H}^1(\Omega;\mathbb{R}^n),\quad \|\boldsymbol{\psi}\|_{\mathbf{H}^1(\Omega;\mathbb{R}^n)} \leq d\left(\|\nabla\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega;\mathbb{M}^n)} + \left|\int_\Omega \boldsymbol{\psi}\,dx\right|\right)$$

Let $\mathbf{b} \overset{\text{def}}{=} \left(\int_\Omega dx\right)^{-1}\int_\Omega (\boldsymbol{\phi} - \mathbf{R}\Theta)\,dx = \mathbf{b}(\boldsymbol{\phi},\Theta)$ so that

$$\boldsymbol{\psi} \overset{\text{def}}{=} \boldsymbol{\phi} - (\mathbf{b} + \mathbf{R}\Theta) \text{ satisfies } \int_\Omega \boldsymbol{\psi}\,dx = \mathbf{0}$$

Combine Poincaré inequality applied to $\boldsymbol{\psi}$ and **(**)** $\square$

Lemma 2: "Geometric rigidity"

G. Friesecke, R.D. James, S. Müller, Comm. Pure Appl. Math. (2002)

$\Omega \subset \mathbb{R}^n$ bounded, connected, open with $\partial\Omega$ Lipschitz-cont's

$\exists$ constant $\Lambda(\Omega)$ s.t. $\forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega;\mathbb{R}^n)$, $\exists\, \mathbf{R} = \mathbf{R}(\boldsymbol{\phi}) \in \mathbb{O}^n_+$ s.t.

$$\boxed{\|\nabla\boldsymbol{\phi} - \mathbf{R}\|_{\mathbf{L}^2(\Omega;\mathbb{M}^n)} \leq \Lambda(\Omega)\,\underbrace{\|\mathrm{dist}(\nabla\boldsymbol{\phi},\mathbb{O}^n_+)\|_{L^2(\Omega)}}_{\overset{\text{def}}{=}\left\{\int_\Omega \inf_{\mathbf{Q}\in\mathbb{O}^n_+} |\nabla\boldsymbol{\phi}(x) - \mathbf{Q}|^2\,dx\right\}^{1/2}}}\quad \square$$

Lemma 3: "A matrix inequality" Let $\mathbf{F} \in \mathbb{M}^n$ s.t. $\det \mathbf{F} > 0$

Then $\boxed{\operatorname{dist}(\mathbf{F}, \mathbb{O}^n_+) \overset{\text{def}}{=} \inf_{\mathbf{Q} \in \mathbb{O}^n_+} |\mathbf{F} - \mathbf{Q}| \leq |\mathbf{F}^T\mathbf{F} - \mathbf{I}|^{1/2}}$

Pf. $\mathbf{Q}_\mathbf{F} \overset{\text{def}}{=} \mathbf{F}(\mathbf{F}^T\mathbf{F})^{-1/2} \in \mathbb{O}^n_+$

$$\operatorname{dist}(\mathbf{F}, \mathbb{O}^n_+) \leq |\mathbf{F} - \mathbf{Q}_\mathbf{F}| = |\overbrace{\mathbf{Q}_\mathbf{F}(\mathbf{F}^T\mathbf{F})^{1/2}}^{=\mathbf{F}} - \mathbf{Q}_\mathbf{F}| \overset{\text{rotational invariance}}{=} |(\mathbf{F}^T\mathbf{F})^{1/2} - \mathbf{I}|$$

$$= \max\{|v_1 - 1|, |v_n - 1|\} \leq \max\{|v_1^2 - 1|^{1/2}, |v_n^2 - 1|^{1/2}\} = |\mathbf{F}^T\mathbf{F} - \mathbf{I}|^{1/2}$$

$0 < v_1 \leq \cdots \leq v_n$: singular values of $\mathbf{F}$ $\square$

Lemma 4: "Ineq. (**) holds if $\Theta = \mathbf{id}$" $\therefore$ $\nabla\Theta = \mathbf{I}$ & $\nabla\Theta^T\nabla\Theta = \mathbf{I}$

$\Omega \subset \mathbb{R}^n$ bounded, connected, open with $\partial\Omega$ Lipschitz-cont's

$\exists$ constant $\Lambda(\Omega)$ s.t. $\forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega; \mathbb{R}^n)$ s.t. $\det \nabla\boldsymbol{\phi} > 0$ a.e. in Ω,

$\boxed{\exists \mathbf{R} = \mathbf{R}(\boldsymbol{\phi}) \in \mathbb{O}^n_+ \text{ s.t. } \|\nabla\boldsymbol{\phi} - \mathbf{R}\|_{\mathbf{L}^2(\Omega; \mathbb{M}^n)} \leq \Lambda(\Omega) \|\nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi} - \mathbf{I}\|^{1/2}_{\mathbf{L}^1(\Omega; \mathbb{S}^n)}}$

Pf. $\mathbf{L2} \Rightarrow \forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega; \mathbb{R}^n)$,

$$\exists \mathbf{R} = \mathbf{R}(\boldsymbol{\phi}) \in \mathbb{O}^n_+ \text{ s.t. } \|\nabla\boldsymbol{\phi} - \mathbf{R}\|_{\mathbf{L}^2(\Omega; \mathbb{M}^n)} \leq \Lambda(\Omega) \|\operatorname{dist}(\nabla\boldsymbol{\phi}, \mathbb{O}^n_+)\|_{L^2(\Omega)}$$

$\mathbf{L3} \Rightarrow \forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega; \mathbb{R}^n)$ s.t. $\det \nabla\boldsymbol{\phi} > 0$ a.e. in Ω,

$$\operatorname{dist}(\nabla\boldsymbol{\phi}(x), \mathbb{O}^n_+)^2 \leq |\nabla\boldsymbol{\phi}(x)^T\nabla\boldsymbol{\phi}(x) - \mathbf{I}| \text{ for almost all } x \in \Omega$$

$\therefore$ integrate over Ω. $\square$

Lemma 5: "Ineq. (**) holds if Θ is injective in $\overline{\Omega}$"

Let $\Theta \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^n)$ be s.t. $\det \nabla\Theta > 0$ in $\overline{\Omega}$ & Θ **injective** in $\overline{\Omega}$. Then inequality (**) of **L1** holds.

Pf. (i) Assumptions made on Θ imply:

$$\begin{cases} \hat{\Omega} \overset{\text{def}}{=} \Theta(\Omega): \text{bounded, connected, open with } \partial\hat{\Omega} \text{ Lipschitz-cont's} \\ \hat{\Theta} \overset{\text{def}}{=} \Theta^{-1} \in \mathcal{C}^1(\{\hat{\Omega}\}^-;\mathbb{R}^n); \forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega;\mathbb{R}^n) \text{ with } \det\nabla\boldsymbol{\phi} > 0 \text{ a.e. in } \Omega, \\ \qquad \hat{\boldsymbol{\phi}} \overset{\text{def}}{=} \boldsymbol{\phi}\circ\hat{\Theta} \in \mathbf{H}^1(\hat{\Omega};\mathbb{R}^n) \ \& \ \det\hat{\nabla}\hat{\boldsymbol{\phi}} > 0 \text{ a.e. in } \hat{\Omega} \end{cases}$$

$L4 \Rightarrow \exists\, c_0(\Theta) \overset{\text{def}}{=} \Lambda(\hat{\Omega})$ s.t. $\forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega;\mathbb{R}^n)$ with $\det\nabla\boldsymbol{\phi} > 0$ a.e. in Ω,

$$\boxed{\begin{gathered}\forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega;\mathbb{R}^n) \text{ with } \det\nabla\boldsymbol{\phi} > 0 \text{ a.e. in } \Omega, \exists \mathbf{R} = \mathbf{R}(\hat{\boldsymbol{\phi}}) = \mathbf{R}(\boldsymbol{\phi},\Theta) \in \mathbb{O}^n_+ \\ \text{s.t. } \|\hat{\nabla}\hat{\boldsymbol{\phi}} - \mathbf{R}\|_{\mathbf{L}^2(\hat{\Omega};\mathbb{M}^n)} \leq c_0(\Theta)\, \|\hat{\nabla}\hat{\boldsymbol{\phi}}^T\hat{\nabla}\hat{\boldsymbol{\phi}} - \mathbf{I}\|^{1/2}_{\mathbf{L}^1(\hat{\Omega};\mathbb{S}^n)}\end{gathered}}$$

(ii)

$$\|\hat{\nabla}\hat{\boldsymbol{\phi}} - \mathbf{R}\|^2_{\mathbf{L}^2(\hat{\Omega};\mathbb{M}^n)} = \int_{\hat{\Omega}} |\hat{\nabla}\hat{\boldsymbol{\phi}} - \mathbf{R}|^2 d\hat{x} = \int_\Omega |\nabla\boldsymbol{\phi}\nabla\Theta^{-1} - \mathbf{R}|^2 \det\nabla\Theta\, dx$$

$$\geq \int_\Omega |\nabla\boldsymbol{\phi} - \mathbf{R}\nabla\Theta|^2 \underbrace{|\nabla\Theta|^{-2}\det\nabla\Theta}_{\geq c_1(\Theta) > 0 \text{ on } \overline{\Omega}} dx$$

(iii)

$$\|\hat{\nabla}\hat{\boldsymbol{\phi}}^T\hat{\nabla}\hat{\boldsymbol{\phi}} - \mathbf{I}\|_{\mathbf{L}^1(\hat{\Omega};\mathbb{S}^n)} = \int_{\hat{\Omega}} |\hat{\nabla}\hat{\boldsymbol{\phi}}^T\hat{\nabla}\hat{\boldsymbol{\phi}} - \mathbf{I}|\, d\hat{x}$$

$$= \int_\Omega |\nabla\Theta^{-T}(\nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi} - \nabla\Theta^T\nabla\Theta)\nabla\Theta^{-1}|\det\nabla\Theta\, dx$$

$$\leq \int_\Omega |\nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi} - \nabla\Theta^T\nabla\Theta| \underbrace{|\nabla\Theta^{-T}|\,|\nabla\Theta^{-1}|\det\nabla\Theta}_{\leq c_2(\Theta) < +\infty \text{ on } \overline{\Omega}} dx$$

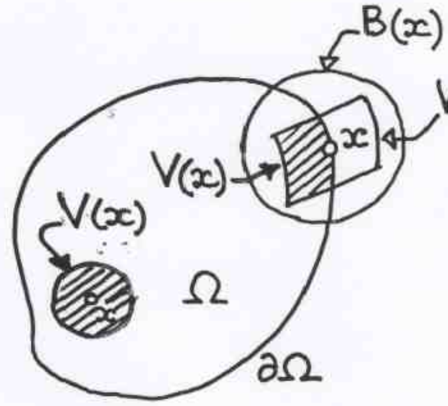
(iv) Hence

$$(**) \quad \boxed{\|\nabla\boldsymbol{\phi} - \mathbf{R}\nabla\Theta\|_{\mathbf{L}^2(\Omega;\mathbb{S}^n)} \leq \underbrace{c(\Theta)}_{= c_0(\Theta)c_1(\Theta)^{-1/2}c_2(\Theta)^{1/2}} \|\nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi} - \nabla\Theta^T\nabla\Theta\|^{1/2}_{\mathbf{L}^1(\Omega;\mathbb{S}^n)}} \quad \square$$

Rk. Ass'ns $\Theta \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^n)$ & $\det\nabla\Theta > 0$ in $\overline{\Omega}$ are essential!

Lemma 6: "Ineq. $(**)$ holds for any $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^n)$ s.t. $\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{\Omega}$"

Pf. (i) $\forall x \in \Omega$, $\exists$ open ball $V(x)$ centered at x s.t. $\boldsymbol{\Theta}|_{\overline{V(x)}}$ is injective



$\forall x \in \partial\Omega$, $\exists$ open ball $B(x)$ s.t.

$(\mathrm{Ext}_{\mathbb{R}^n}\boldsymbol{\Theta})|_{\overline{B(x)}}$ is injective ($\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{\Omega}$)

$\partial\Omega$ Lipschitz-cont's $\Rightarrow \exists\, W(x)$ open nbhd of x def'd by a local frame s.t. $W(x) \subset B(x)$ and $V(x) \overset{\text{def}}{=} W(x) \cap \Omega$ is connected and $\partial V(x)$ is Lipschitz-cont's.

By compactness, $\Omega = \bigcup_{j=1}^{N} V(x_j)$. By re-ordering $V_j = V(x_{\sigma(j)})$:

$\Omega = \bigcup_{j=1}^{N} V_j$, V_j bounded, connected, open, ∂V_j Lipschitz-cont's

$U_k \overset{\text{def}}{=} \bigcup_{j=1}^{k} V_j$ connected, $1 \le k \le N$, and $\boldsymbol{\Theta}|_{\overline{V_j}}$ injective, $1 \le j \le N$.

(ii) Induction: For some $1 \le k < N$, $\exists\, C_k(\boldsymbol{\Theta})$, $\exists\, \mathbf{R}_k = \mathbf{R}_k(\boldsymbol{\Phi},\boldsymbol{\Theta}) \in \mathbb{O}^n_+$ s.t.

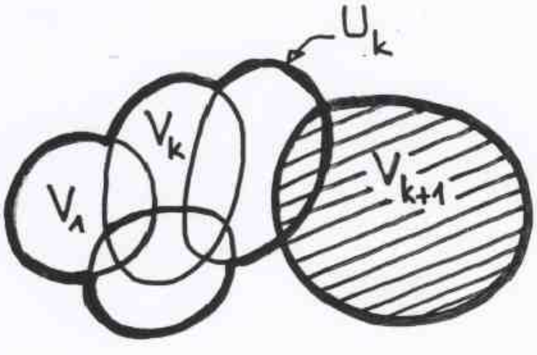
$$(**)_k \quad \|\boldsymbol{\nabla\Phi}_k - \mathbf{R}_k \boldsymbol{\nabla\Theta}_k\|_{\mathbf{L}^2(U_k;\mathbb{M}^n)} \le C_k(\boldsymbol{\Theta}) \, \|\boldsymbol{\nabla\Phi}_k^T \boldsymbol{\nabla\Phi}_k - \boldsymbol{\nabla\Theta}_k^T \boldsymbol{\nabla\Theta}_k\|^{1/2}_{\mathbf{L}^1(U_k;\mathbb{S}^n)}$$

$\boldsymbol{\Phi}_k \overset{\text{def}}{=} \boldsymbol{\Phi}|_{U_k}$, $\boldsymbol{\Theta}_k \overset{\text{def}}{=} \boldsymbol{\Theta}|_{\overline{U}_k}$ (Note: $U_1 = V_1$)

L5 $\Rightarrow \exists\, \tilde{C}_k(\boldsymbol{\Theta})$, $\exists\, \tilde{\mathbf{R}}_{k+1} = \tilde{\mathbf{R}}_{k+1}(\boldsymbol{\Phi},\boldsymbol{\Theta}) \in \mathbb{O}^n_+$ s.t.

$$(**)'_k \quad \|\boldsymbol{\nabla}\tilde{\boldsymbol{\Phi}}_{k+1} - \tilde{\mathbf{R}}_{k+1} \boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}}_{k+1}\|_{\mathbf{L}^2(V_{k+1};\mathbb{M}^n)} \le \tilde{C}_k(\boldsymbol{\Theta}) \, \|\boldsymbol{\nabla}\tilde{\boldsymbol{\Phi}}_{k+1}^T \boldsymbol{\nabla}\tilde{\boldsymbol{\Phi}}_{k+1} - \boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}}_{k+1}^T \boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}}_{k+1}\|^{1/2}_{\mathbf{L}^1(V_{k+1};\mathbb{S}^n)}$$

$\tilde{\boldsymbol{\Phi}}_{k+1} \overset{\text{def}}{=} \boldsymbol{\Phi}|_{V_{k+1}}$, $\tilde{\boldsymbol{\Theta}}_{k+1} \overset{\text{def}}{=} \boldsymbol{\Theta}|_{\overline{V}_{k+1}}$



By means of a technical trick, inequalities $(**)_k$ & $(**)'_k$ yield a single inequality $(**)_{k+1}$ over the set U_{k+1}. $\square$

Another approach to the fundamental theorem of Riemannian geometry in $\mathbb{R}^3$, by way of rotation fields:
P.G. Ciarlet, L. Gratie, O. Iosifescu, C. Mardare, C. Vallée, JMPA [2007]

THM $\Omega \subset \mathbb{R}^3$: simply-connected, open
$\mathbf{C} \in \mathbf{W}^{1,\infty}_{loc}(\Omega; \mathbb{S}^3_>)$ given that satisfies

$$(*) \quad \boxed{\mathbf{CURL}\, \Lambda + \mathbf{COF}\, \Lambda = 0 \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^3)},$$

where

$$\Lambda \overset{\text{def}}{=} \frac{1}{\det \mathbf{U}} \left\{ \mathbf{U}(\mathbf{CURL}\,\mathbf{U})^T \mathbf{U} - \frac{1}{2}\left(\mathrm{tr}\,[\mathbf{U}(\mathbf{CURL}\,\mathbf{U})^T]\right)\mathbf{U} \right\}$$
$$\mathbf{U} \overset{\text{def}}{=} \mathbf{C}^{1/2}$$

Then there exists $\boldsymbol{\Theta} \in \mathbf{W}^{2,\infty}_{loc}(\Omega; \mathbb{R}^3)$ such that

$$\boxed{\nabla \boldsymbol{\Theta}^T \nabla \boldsymbol{\Theta} = \mathbf{C} \text{ in } \mathbf{W}^{1,\infty}_{loc}(\Omega; \mathbb{S}^3_>)}$$

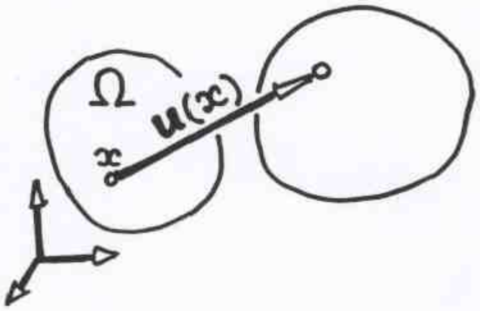
Necessity of $(*)$: C. Vallée (1992)
No Christoffel symbols; "intrinsic" matrix operators
Proof consists in:
(a) Determining an orthogonal matrix field $\mathbf{R} \in \mathbf{W}^{1,\infty}_{loc}(\Omega; \mathbb{O}^3)$ (an idea that goes back to R.T. Shield (1973));
(b) Determining an immersion $\boldsymbol{\Theta} \in \mathbf{W}^{2,\infty}_{loc}(\Omega; \mathbb{R}^3)$ that satisfies $\nabla \boldsymbol{\Theta} = \mathbf{R}\mathbf{C}^{1/2}$ in Ω;
by successively solving two Pfaff systems.

For (a): $\partial_j \mathbf{R} = \mathbf{R}\mathbf{A}_j$, where

$$\mathbf{A}_j := \frac{1}{2}\left\{ \mathbf{U}^{-1}(\nabla \mathbf{c}_j - (\nabla \mathbf{c}_j)^T)\mathbf{U}^{-1} + \mathbf{U}^{-1}\partial_j \mathbf{U} - (\partial_j \mathbf{U})\mathbf{U}^{-1} \right\},$$
$\mathbf{c}_j :=$ j-th column of $\mathbf{U}^2 = \mathbf{C}$

7. Intrinsic approach to 3D-Linearized Elasticity

Pure traction pbm:



$$\boxed{\begin{aligned} J(\mathbf{u}) &= \inf_{\mathbf{v}\in \mathbf{H}^1(\Omega;\mathbb{R}^3)} J(\mathbf{v}) \\ J(\mathbf{v}) &= \frac{1}{2}\int_\Omega A^{ijk\ell} e_{k\ell}(\mathbf{v})\overbrace{e_{ij}(\mathbf{v})}^{=\frac{1}{2}(\partial_j v_i+\partial_i v_j)}dx - \underbrace{\int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx}_{L(\mathbf{v})}\end{aligned}}$$

$L(\mathbf{v})=0\ \forall \mathbf{v}\in\mathbf{H}^1(\Omega;\mathbb{R}^3)$ s.t. $e_{ij}(\mathbf{v})=0 \Leftrightarrow \forall \underbrace{\mathbf{v}=\mathbf{a}+\mathbf{b}\wedge\mathbf{id}}$ with $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$ } if Ω connected

Existence: Korn's inequality; Uniqueness up to

Theorem: Ω bdd, simply-connected, connected; Lipschitz bdry

Given: $\mathbf{e}=(e_{ij})\in\mathbf{L}^2_{sym}(\Omega)$ s.t. (weak <u>St Venant conditions</u>)

$$\boxed{\mathcal{R}_{ijk\ell}(\mathbf{e})\overset{\text{def}}{=}\partial_{\ell j}e_{ik}+\partial_{ki}e_{j\ell}-\partial_{\ell i}e_{jk}-\partial_{kj}e_{i\ell}=0 \text{ in } H^{-2}(\Omega)}$$

Then there exists $\mathbf{v}\in\mathbf{H}^1(\Omega;\mathbb{R}^3)$ s.t. $\boxed{e_{ij}=\frac{1}{2}(\partial_j v_i+\partial_i v_j)}$

Uniqueness: up to $\mathbf{a}+\mathbf{b}\wedge\mathbf{id}$

Proof is based on a "weak Poincaré Lemma"; see p. 12 bis

Corollary: There exists an isomorphism

$$\boxed{\mathcal{F}: \mathbf{E}(\Omega)\overset{\text{def}}{=}\{\mathbf{e}\in\mathbf{L}^2_{sym}(\Omega);\ \mathcal{R}_{ijk\ell}(\mathbf{e})=0\}\to\mathbf{H}^1(\Omega;\mathbb{R}^3)/\{\mathbf{v}=\mathbf{a}+\mathbf{b}\wedge\mathbf{id}\}}$$

Rk: This corollary provides a new pf of Korn's inequality

P.G. Ciarlet, P. Ciarlet, Jr., M3AS (2005)
P.G. Ciarlet, P. Ciarlet, Jr., G. Geymonat, F. Krasucki, C.R.A.S. (2007)
C. Amrouche, P.G. Ciarlet, P.G. Ciarlet, Jr., C.R.A.S. (2007)

Theorem: Ω bdd, simply-connected, connected; Lipschitz bdry

The minimization pbm: Find $\boldsymbol{\varepsilon}\in\mathbf{E}(\Omega)$ such that

$$\boxed{j(\boldsymbol{\varepsilon})=\inf_{\mathbf{e}\in\mathbf{E}(\Omega)} j(\mathbf{e}),\qquad j(\mathbf{e})\overset{\text{def}}{=}\frac{1}{2}\int_\Omega A^{ijk\ell}e_{k\ell}e_{ij}\,dx-(L\circ\mathcal{F})(\mathbf{e})}$$

has one and only one solution $\boldsymbol{\varepsilon}$. Besides, $\boldsymbol{\varepsilon}=\mathbf{e}(\mathbf{u})$

The <u>stress</u> $\sigma=(\sigma^{ij})$ is then given by $\boxed{\sigma^{ij}=A^{ijkl}\varepsilon_{kl}}$

Weak version of a classical theorem of Poincaré

P.G. Ciarlet, P. Ciarlet, Jr., M3AS (2005)

A classical *theorem of Poincaré* (see, e.g., page 235 in Schwartz[14]) asserts that, if functions $h_k \in \mathcal{C}^1(\Omega)$ satisfy $\partial_l h_k = \partial_k h_l$ in a simply-connected open subset Ω of $\mathbb{R}^3$ (or $\mathbb{R}^n$ for that matter), then there exists a function $p \in \mathcal{C}^2(\Omega)$ such that $h_k = \partial_k p$ in Ω. This theorem was extended by Girault & Raviart[12] (see Theorem 2.9 in Chapter 1), who showed that, if functions $h_k \in L^2(\Omega)$ satisfy $\partial_l h_k = \partial_k h_l$ in $H^{-1}(\Omega)$ on a bounded, connected and simply-connected open subset Ω of $\mathbb{R}^3$ with a Lipschitz-continuous boundary, then there exists $p \in H^1(\Omega)$ such that $h_k = \partial_k p$ in $L^2(\Omega)$. We now carry out this extension one step further.

Theorem .. *Let Ω be a bounded, connected, and simply-connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Let $h_k \in H^{-1}(\Omega)$ be distributions that satisfy*

$$\boxed{\partial_l h_k = \partial_k h_l \text{ in } H^{-2}(\Omega).}$$

Then there exists a function $p \in L^2(\Omega)$, unique up to an additive constant, such that

$$\boxed{h_k = \partial_k p \text{ in } H^{-1}(\Omega).}$$

THEOREM: Ω bdd, connected; Lipschitz bdry

(1) Given $\mathbf{e} = (e_{ij}) \in \mathbf{L}^2_{sym}(\Omega)$ s.t. (<u>Donati conditions #1</u>):

$$\boxed{\int_\Omega e_{ij} s_{ij}\, dx = 0 \quad \text{for all } \mathbf{s} = (s_{ij}) \in \mathbf{H}^1_{0,sym}(\Omega) \text{ s.t. } \mathbf{div}\, \mathbf{s} = \mathbf{0} \text{ in } \Omega}$$

Then there exists $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$ s.t. $\boxed{e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)}$

(2) Given $\mathbf{e} = (e_{ij}) \in \mathbf{L}^2_{sym}(\Omega)$ s.t. (Donati conditions #2):

$$\boxed{\int_\Omega e_{ij} s_{ij}\, dx = 0 \quad \text{for all } \mathbf{s} = (s_{ij}) \in \mathbf{L}^2_{sym}(\Omega) \text{ s.t. } \mathbf{div}\, \mathbf{s} = \mathbf{0} \text{ in } \mathbf{H}^{-1}(\Omega)}$$

Then there exists $\mathbf{v} = (v_i) \in \mathbf{H}^1_0(\Omega)$ s.t. $\boxed{e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)}$

T. W. Ting, Tensor, N.S. (1974)

G. Geymonat, F. Krasucki, Rend. Accad. Naz. Sci. (2005)

C. Amrouche, P.G. Ciarlet, L. Gratie, S. Kesavan, JMPA (2006)

Both (1) and (2) again lead to minimization problems that **directly provide** the **strain** $\boldsymbol{\varepsilon} = (\varepsilon_{kl})$

$\therefore$ that **directly provide** the **stresses** $\boxed{\sigma^{ij} = A^{ijkl}\varepsilon_{kl}}$

FINITE ELEMENT APPROXIMATION:

P. G. Ciarlet, P. Ciarlet, Jr., S. Sauter, Jun Zou (in progress)