Geometry of Curves in Metric Spaces of Curvature Bounded Above

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0. Background

☞ 0.1. Metric Spaces of Curvature Bounded above

- A.D. Alexandrov
- Ideas: A generalization from Riemannian manifolds of curvature bounded above

Def [0.1] X is *CAT(K)* (also R_K) if

- any pair of its points lie on a unique geodesic, and
- any of its triangles is CAT(K).

Def [0.2] X has *curvature bounded above by K* if every point of X lies in a CAT(K) neighborhood.

Thm [0.3] [Reshetnyak] If the length of a rectifiable closed curve in a CAT(K) space is less than $\frac{2\pi}{\sqrt{K}}$ then there is a convex domain in S_K that majorizes it.

0.2. Angles and Directions

Def [0.4] Fix K. An *angle* $\angle_p(\alpha, \beta)$ between constant-speed curves α and β in X defined on an interval $[a, b]$ with a common left endpoint p is the limit $\lim_{t_1, t_2 \to a^+} \bar{\angle}_{\bar{p}} \left(\overline{\bar{p}\,\alpha(\bar{t}_1)}, \overline{\bar{p}\,\beta(\bar{t}_2)}\right)$, if it exists, where $\overline{AB}$ denotes the minimizing geodesic joining points A and B in S_K and $\bar{\angle}$ is the usual angle in S_K.

⋆ Note: This is independent of K.

Def [0.5] A curve $\gamma : [a, b] \to X$ in X has a *right* (respectively, *left*) *direction* at a point $\gamma(t)$ if an angle between $\gamma|_{[t,b]}$ (respectively, $\gamma|_{[a,t]}$) and itself exists (and hence is zero). Two curves with a common left (right) endpoint p, both having right (left) directions at p have *the same right (left) direction* at p if the angle between them is zero.

Thm [0.6] [Alexandrov] In a CAT(K) space X, the angle between any two geodesics at their common endpoints exists. If α_1, α_2 and α_3 are angles of a triangle in X corresponding to angles $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$ and $\widetilde{\alpha}_3$ of a triangle in S_K with the same sidelengths, then $\alpha_i \leq \widetilde{\alpha}_i$ for $i = 1, 2, 3$. An equality holds for some i if and only if the two triangles bound totally geodesic surfaces isometric to each other.

Thm [0.7] [Ballman] Let x_n, y_n and z_n be sequences of points in an R_K space X. If $x_n \to x$, $y_n \to y$ and $z_n \to z$ with $y \in X$ and $x, z \neq y$ then for sufficiently large n the angles $\angle x_n y_n z_n$ and $\angle x_n y z_n$ are defined, and

$$\limsup_{n\to\infty} \angle x_n y_n z_n \leq \lim_{n\to\infty} \angle x_n y z_n = \angle xyz.$$

0.3. Metrics of CAT(0) Spaces

Thm [0.8] [Bridson and Haefliger] In a CAT(0) space (X, d), any pair of constant-speed geodesics c_1 and c_2 parametrized by $[0, 1]$ satisfy

$$r_t \leq (1-t)r_0 + tr_1$$

for all $t \in [0, 1]$, where r_t is the distances between $c_0(t)$ and $c_1(t)$.

1. Total Curvature of Curves

☞ 1.1. Total Rotation

Def [1.1] The *total rotation* of a polysegment σ is the sum

$$\kappa^*(\sigma) = \sum_{i=1}^{k-1} (\pi - \widehat{p}_i),$$

where p_i's are the vertices of σ and each $\widehat{p}_i$ is the angle between the left and right directions of σ at p_i

1.2. Total Curvature

- Generalization of the classical one.

Def [1.2] The *total curvature* $\kappa(\gamma)$ of a curve γ is

$$\kappa(\gamma) = \limsup_{\mu_\gamma(\sigma)\to 0} \kappa^*(\sigma) = \lim_{\epsilon\to 0^+} \sup_{\sigma\in\Sigma_\epsilon(\gamma)} \kappa^*(\sigma),$$

where for each $\epsilon > 0$, $\Sigma_\epsilon(\gamma)$ is the set of polysegments σ inscribed in γ such that $\mu_\gamma(\sigma) < \epsilon$. Here $\mu_\gamma(\sigma)$ is the maximum of the diameters of subarcs of γ subdivided by the vertices of σ.

Thm [1.3] For a polysegment in an R_K space, its total curvature and its total rotation coincide.

Thm [1.4] Let τ_n be any sequence of polysegments inscribed in a curve γ in an R_K space such that $\mu_\gamma(\tau_n) \to 0$. Then $\kappa(\tau_n) \to \kappa(\gamma)$. Furthermore, if $\kappa(\gamma)$ is finite then γ is rectifiable.

1.3. Lower-Semi Continuity of Total Curvature

Thm [1.5] If a sequence of curves γ_m converges to a curve γ in a CAT(K) space, then $\kappa(\gamma) = \liminf_{m\to\infty} \kappa(\gamma_m)$.

2. Metrics of CAT(K) Spaces, with $K > 0$

☞ 2.1. Local Quadratic Subconvexity of Metrics

Fix $K > 0$. Let c_0 and c_1 be a pair of constant-speed minimizing geodesics linearly parametrized by $[0, 1]$ in a CAT(K) space (X, d). Let r_t be the distances between $c_0(t)$ and $c_1(t)$. Then we have the following

Thm [2.1] The inequalities

$$\sin\frac{r_t\sqrt{K}}{2} \le \cos\frac{\pi t}{2}\sin\frac{r_0\sqrt{K}}{2} + \sin\frac{\pi t}{2}\sin\frac{r_1\sqrt{K}}{2}$$

and

$$r_t \le \left(\cos\frac{\pi t}{2}\right) r_0 + \left(\sin\frac{\pi t}{2}\right) r_1$$

hold for all $t \in [0,1]$, provided that the two geodesics are not too long and not too far apart. Thus these inequalities hold *locally*.

3. Length Estimates

3.1. Chord-Curvature Estimate

Thm [3.1] Let γ be a curve in a CAT(K) space, s its arclength, r its chordlength and κ its total curvature. Assume that $s < \frac{\pi}{\sqrt{K}}$ and that $\kappa < \pi$ if $K < 0$ and $\kappa + \lambda r < \pi$ if $K > 0$. Then $s \leq s(r, \kappa)$, where $s(r,\kappa) = \dfrac{2}{\lambda}\zeta^{-1}\left(\dfrac{\zeta\left(\frac{r\lambda}{2}\right)}{\cos\frac{r}{2}}\right)$.

Here and below, $\lambda = \sqrt{|K|}$ and ζ is the sin function if $K > 0$ and the sinh function if $K < 0$.

3.2. Circumradius-Curvature Estimate

Thm [3.2] Let γ be a curve contained in a closed ball of radius R in an R_K space, where $K \neq 0$ and $R < \frac{\zeta^{-1}(1)}{\lambda}$. Then $\ell(\gamma) \leq S(R, \kappa(\gamma))$, where $S(R, \kappa)$ is the length of a certain isosceles bisegment of the same total curvature, possibly with an inserted circular arc, inscribed in a closed disk of the same radius in S_K.

Thm [3.3] Let γ be a curve at a uniform distance $R < \frac{\pi}{2\sqrt{K}}$ from a fixed point in a CAT(K) space with endpoints a and b, and γ' a circular arc of radius R in S_K with endpoints a' and b'. If $\ell(\gamma) \geq \ell(\gamma')$, then $\kappa(\gamma) \geq \kappa(\gamma')$, and if $d(a,b) \geq d'(a',b')$, then $\ell(\gamma) \geq \ell(\gamma')$.

Here d and d' are metrics in the CAT(K) space and in S_K, respectively.

4. Selected References

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Def [4.1] A triangle in X is *$CAT(K)$* if

- its *perimeter*, i.e., the sum of its sidelengths, is no greater than $\frac{2\pi}{\sqrt{K}}$, and
- the distance between any pair of points on the sides of the triangle is no greater than the distance between the corresponding points on its comparison triangle in S_K.

Def [4.2] A *nonexpanding* map is a map between metric spaces that never increases the distance between points. A convex domain D in S_K *majorizes* a rectifiable closed curve γ in X if a nonexpanding map exists from D to X, with its restriction to the boundary ∂D of D an arclength preserving map onto the image of γ. We also say that the ∂D *majorizes* γ.

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