A Fractional Geometric BM with Jump and Stochastic volatility model

Prof. Dr. Pairote Sattayatham

Risk and control Lab,

Suranaree University of Technology

Part I: Main Idea

We would liked to extend Merton's firm value model of the form

$$dS_t = \mu S_t dt + \nu S_t dW_t, \quad 0 \le t \le T$$

into the follow fractional model with compound Poisson jump and with stochastic volatility model.

$$dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dB_t + S_{t-} Y_t dN_t \qquad (1)$$

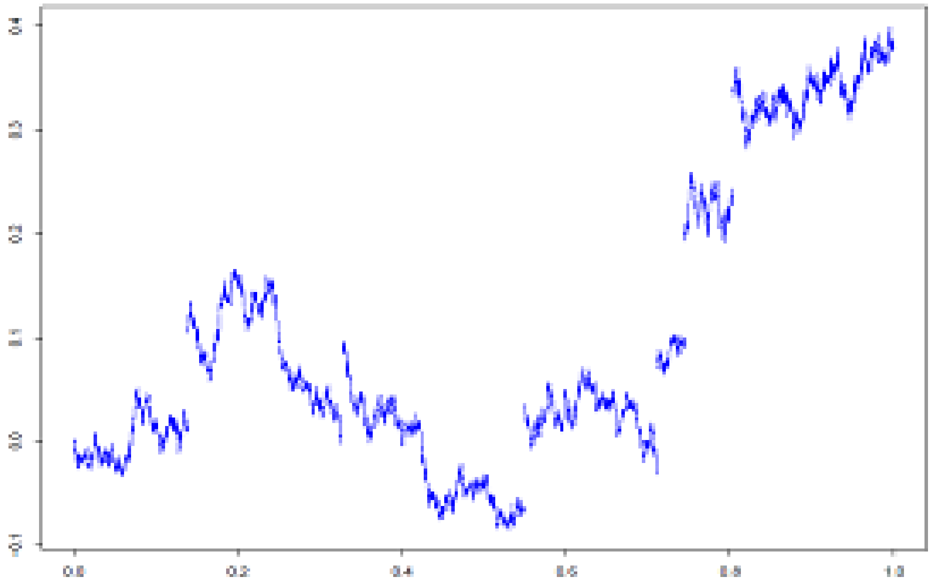
$$dv_t = (\omega - \theta v_t) dt + \xi v_t d\overline{B}_t \qquad (2)$$

The reason is that the asset price of a firm exhibit

1. long range dependence with $H = 0.6$ and
2. jump from the following observation

A closer look at real-life data

- The stock price process exhibit jump



the process B_t in eqn(1) is defined by

$$B_t = \int_0^t (t-s)^\alpha dW_s$$

This process is not a semi martingale so we cannot define stocastic integral dB_t in the Ito's sense. However, one can show that

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s$$

converges to B_t in $L_2(\Omega)$ and uniformly on $t \in [0,T]$. Hence, in order to study eqn(1), we shall consider the following approximate model

$$dS_t^{\varepsilon} = \mu_t S_t^{\varepsilon} dt + \sqrt{v_t^{\varepsilon}} S_t^{\varepsilon} dB_t^{\varepsilon} + S_{t-}^{\varepsilon} Y_t dN_t \qquad (3)$$

$$dv_t^{\varepsilon} = (\omega - \theta v_t^{\varepsilon}) dt + \xi v_t^{\varepsilon} d\overline{B}_t^{\varepsilon} \qquad (4)$$

These equation are well defined in the Ito's sense. The limit of the solutions of (3),(4) in $L_2(\Omega)$ as $\varepsilon \to 0$ and uniformly on t will be the solution of (1) and (2).

Part II: Results

Solutions of the approximate models:

Theorem 1 By using Ito's formula for jump process, the solution of (3) is

$$S_t^\varepsilon = S_0 \exp\left[-\frac{1}{2}\varepsilon^{2\alpha} t + \int_0^t H_s^\varepsilon ds + \varepsilon^\alpha \int_0^t \sqrt{v_\varepsilon^s} dW_s + \int_0^t \log(1+Y_s) dN_s \right] \quad (5)$$

where

$$\int_0^t H_s^\varepsilon ds = \mu_t t + \alpha \int_0^t \sqrt{v_s^\varepsilon} \varphi_s^\varepsilon ds \quad \text{and} \quad \varphi_s^\varepsilon = \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s$$

Convergence of the approximate model

Define a proces S_t^* as follows:

$$S_t^* = S_0 \exp\left[\mu_t t + \alpha \int_0^t \sqrt{v_s} \varphi_s ds + \varepsilon^{\alpha} \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s) dN_s \right] (6)$$

With some assumptions about the initial condition, we can prove the following theorem.

Theorem 2. $S_t^{\varepsilon} \to S_t^*$ as $\varepsilon \to 0$ and uniformly for $t \in [0,T]$.

Proof. The proof is more technical using tools from functional analysis and stochastic analysis.

Part III Simulation for testing the model

Part IV Application to credit risk

Recall that for the Merton model with the asset price process of the form

$$dS_t = \mu S_t dt + \nu S_t dW_t, \quad 0 \le t \le T$$

$$P(V_T \le D) = \phi\left(\frac{\ln(D/V_0) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$