Constants and Normal Structure in Banach Spaces

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This talk is based on the following papers:

- A. Jimenez-Melado, E. Llorens-Fuster and S. Saejung, The von Neumann-Jordan constant, weak orthogonality and normal structure in Banach spaces, Proc. Amer. Math. Soc. 134 (2006), no. 2, 355–364.
- S. Saejung, On James and von Neumann-Jordan constants and sufficient conditions for the fixed point property. J. Math. Anal. Appl. 323 (2006), no. 2, 1018-1024.
- S. Saejung, Sufficient conditions for uniform normal structure of Banach spaces and their duals. J. Math. Anal. Appl. 330 (2007), no. 1, 597–604.

- S. Saejung, The characteristic of convexity of a Banach space and normal structure, J. Math. Anal. Appl., 337, (2008) 123-129.
- E. Casini, P. L. Papini, and S. Saejung, Some estimates for the weakly convergent sequence coefficient in Banach spaces, J. Math. Anal. Appl. 346 (2008) 177-182.
- J. Gao and S. Saejung, Normal structure and the generalized James and Zbaganu constants, Nonlinear Analysis Series A: Theory, Methods & Applications, 71(7-8) (2009), 3047-3052.
- J. Gao and S. Saejung, Some geometric measures of spheres in Banach spaces, Appl. Math Comp. 214 (2009) 102-107.

This talk is organized as follows:

- Some definitions and historical remarks
- James and von Neumann–Jordan constants
- parameterized James and von Neumann–Jordan constants
- generalized James and von Neumann–Jordan constants
- Quantitative result

Let X be a Banach space.
The research theme of this talk:

properties or conditions on a Banach space X

$$\Downarrow$$

normal structure of X

Recall that the Banach space X has *normal structure*[1] if every nonempty bounded closed convex subset C of X, with $\operatorname{diam} C > 0$, contains a *non-diametral point*, that is, there exists $x_0 \in C$ such that

$$\sup\{\|x - x_0\| : x \in C\} < \operatorname{diam} C.$$

[1] Brodskiĭ, M. S.; Mil'man, D. P. On the center of a convex set. (Russian) Doklady Akad. Nauk SSSR (N.S.) 59, (1948). 837–840.

Kirk's fixed point theorem[2]

$$X \text{ has normal structure and is reflexive}$$
$$\Rightarrow X \text{ has the fixed point property}$$

Recall that X has the fixed point property if for every bounded closed convex subset C of X and every nonexpansive self-mapping $T\colon C \to C$, that is,

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C,$$

there exists a point $x_0 \in C$ such that

$$x_0 = Tx_0,$$

that is, x_0 is a fixed point of T.

[2] *A fixed point theorem for mappings which do not increase distances.* Amer. Math. Monthly **72** (1965), 1004-1006.

Spaces with/without normal structure

Spaces with normal structure

- Finite dimensional spaces
- Uniformly convex spaces (Clarkson, 1936)
- Uniformly smooth spaces

Spaces without normal structure

- $C[a, b]$ (the space of real-valued continuous functions on $[a, b]$)
- Bynum spaces (Bynum, 1980)

To test whether a given Banach space has normal structure is not an easy task.

Two starting points:

Gao and Lau[3] proved that

$$J(X) < \frac{3}{2} \Rightarrow X \text{ has normal structure.}$$

Kato, Maligranda, and Takahashi[4] proved that

$$C_{\mathrm{NJ}}(X) < \frac{5}{4} \Rightarrow X \text{ has normal structure.}$$

Recall that

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : \|x\| = \|y\| = 1\}$$

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} : \|x\| + \|y\| \neq 0\right\}.$$

[3] On two classes of Banach spaces with uniform normal structure. Studia Math. 99 (1991), no. 1, 41–56.

[4] On James and Jordan-von Neumann constants and the normal structure coefficient of Banach spaces. Studia Math. 144 (2001), no. 3, 275–295.

Some facts on $J(X)$ and $C_{\mathrm{NJ}}(X)$:

- $\sqrt{2} \leq J(X) \leq 2$
- X is a Hilbert space $\Rightarrow J(X) = \sqrt{2}$
- $1 \leq C_{\mathrm{NJ}}(X) \leq 2$
- X is a Hilbert space $\Leftrightarrow C_{\mathrm{NJ}}(X) = 1$
- $\frac{J(X)^2}{2} \leq C_{\mathrm{NJ}}(X) \leq J(X)$
- $J(\ell_p) = J(L_p[a,b]) = 2^{1/p}$ if $1 < p \leq 2$
- $C_{\mathrm{NJ}}(\ell_p) = C_{\mathrm{NJ}}(L_p[a,b]) = 2^{2/p-1}$ if $1 < p \leq 2$

Recall that

- $\ell_p = \{(x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$
- $L_p[a,b] = \{f : f$ is a real-valued function on $[a,b]$ and $\int_{[a,b]} |f|^p d\mu < \infty\}$

Let X be a Banach space.
The research theme of this talk:

$$\text{conditions on a Banach space } X \text{ in terms of } J(X) \text{ or } C_{\mathrm{NJ}}(X)$$
$$\Downarrow$$
$$\text{normal structure of } X$$

The strongest results so far:

$$J(X) < \frac{3}{2} \Rightarrow J(X) < \frac{1+\sqrt{5}}{2} \approx 1.61\text{[5]} \Rightarrow X \text{ has normal structure}$$

$$C_{\mathrm{NJ}}(X) < \frac{5}{4} \Rightarrow C_{\mathrm{NJ}}(X) < \frac{1+\sqrt{3}}{2} \approx 1.36\text{[6]} \Rightarrow X \text{ has normal structure}$$

[5] Dhompongsa, Keawkhao and Tasena, J. Math. Anal. Appl. 285 (2003), no. 2, 419–435.

[6] S. Saejung, J. Math. Anal. Appl. 323 (2006), no. 2, 1018-1024.

The strongest results so far:

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Remark: Both sufficient conditions cannot be applied for ℓ_p or $L_p[a,b]$ where p is near 1. In fact, it is known that all ℓ_p or $L_p[a,b]$ where $1 < p < \infty$ have normal structure and

$$J(\ell_p) = J(L_p[a,b]) = 2^{1/p} \text{ if } 1 < p \leq 2;$$

$$C_{\mathrm{NJ}}(\ell_p) = C_{\mathrm{NJ}}(L_p[a,b]) = 2^{2/p-1} \text{ if } 1 < p \leq 2.$$

[5] Dhompongsa, Keawkhao and Tasena, J. Math. Anal. Appl. 285 (2003), no. 2, 419–435.

[6] S. Saejung, J. Math. Anal. Appl. 323 (2006), no. 2, 1018-1024.

Parameterized James and von Neumann–Joradan constants

We study these constants:[7] for $0 \le t \le 1$

$$J^t(X) = \sup\{\min\{\|x+ty\|, \|x-ty\|\} : \|x\| = \|y\| = 1\}$$

$$C_{\mathrm{NJ}}^t(X) = \frac{1}{2(1+t^2)} \sup\{\|x+ty\|^2 + \|x-ty\|^2 : \|x\| = \|y\| = 1\}.$$

Remark:

- $J(X) = J^1(X)$
- $C_{\mathrm{NJ}}(X) = \sup\{C_{\mathrm{NJ}}^t(X) : 0 \le t \le 1\}$

[7] S. Saejung, Sufficient conditions for uniform normal structure of Banach spaces and their duals. J. Math. Anal. Appl. 330 (2007), no. 1, 597–604.

Better sufficient conditions:

$$J(X) < \frac{1+\sqrt{5}}{2} \quad \Big(\Leftrightarrow J(X) < 1 + \frac{1}{J(X)} \Big)$$

$$\Rightarrow J^t(X) < 1 + \frac{t}{J^t(X) + 1 - t} \text{ for some } 0 \le t \le 1$$

$$\Rightarrow X \text{ has normal structure}$$

Remark: If $X = \ell_p$ or $X = L_p[a,b]$ where $1 < p < \infty$, then $J^t(X) < 1 + \frac{t}{J^t(X)+1-t}$ for some $0 \le t \le 1$.

Better sufficient conditions:

$$C_{\mathrm{NJ}}(X) < \frac{1+\sqrt{3}}{2}$$

$$\Rightarrow (1+t^2)C^t_{\mathrm{NJ}}(X) < \frac{(1+ts)^2}{2(1+s^2)C^s_{\mathrm{NJ}}(X^*) - (1+s)^2}$$

for some $0 \leq t, s \leq 1$

$\Rightarrow X$ has normal structure

Remark: If $X = \ell_p$ or $X = L_p[a,b]$ where $1 < p < \infty$, then $(1+t^2)C^t_{\mathrm{NJ}}(X) < \frac{(1+ts)^2}{2(1+s^2)C^s_{\mathrm{NJ}}(X^*)-(1+s)^2}$ for some $0 \leq t, s \leq 1$.

Some improvement in terms of these constants

In 2006, Jimenez-Melado, Llorens-Fuster and Saejung[8] proved the following:

$$J(X) < 1 + \frac{1}{\mu(X)} \Rightarrow X \text{ has normal structure}$$

$$C_{\mathrm{NJ}}(X) < 1 + \frac{1}{\mu(X)^2} \Rightarrow X \text{ has normal structure}$$

Recall that[9]

$$\mu(X) = \inf \left\{ \begin{array}{c} r > 0 : \limsup_n \|x + x_n\| \leq r \limsup_n \|x - x_n\| \\ \text{for all } x \in X \text{ and all weakly null sequences} \\ \{x_n\} \text{ in } X. \end{array} \right\}.$$

[8]Proc. Amer. Math. Soc. 134 (2006), no. 2, 355–364.

[9]B. Sims, *A class of spaces with weak normal structure*, Bull. Austral. Math. Soc. **50** (1994), 523-528.

$$J(X) < 1 + \frac{1}{\mu(X)} \Rightarrow X \text{ has normal structure}$$

$$C_{\mathrm{NJ}}(X) < 1 + \frac{1}{\mu(X)^2} \Rightarrow X \text{ has normal structure}$$

Note: Both results are sharp in the sense that there is a Banach space X such that X fails to have normal structure and

$$J(X) = 1 + \frac{1}{\mu(X)} \quad \text{and} \quad C_{\mathrm{NJ}}(X) = 1 + \frac{1}{\mu(X)^2}.$$

We can prove the following results[10]:

$$\begin{aligned} J(X) < 1 + \frac{1}{\mu(X)} \\ \Rightarrow J^t(X) < 1 + \frac{t}{\mu(X)} \text{ for some } 0 \le t \le 1 \\ \Rightarrow X \text{ has normal structure} \end{aligned}$$

$$\begin{aligned} C_{\mathrm{NJ}}(X) < 1 + \frac{1}{\mu(X)^2} \\ \Rightarrow C_{\mathrm{NJ}}^t(X) < \frac{\left(1 + \frac{t}{\mu(X)}\right)^2}{1+t^2} \text{ for some } 0 \le t \le 1 \\ \Rightarrow X \text{ has normal structure} \end{aligned}$$

[10] J. Gao and S. Saejung, Appl. Math Comp. 214 (2009) 102-107.

Generalized James and von Neumann–Joradan constants

Let $B_X = \{x \in x : \|x\| \leq 1\}$. Based on the Hexagonal Lemma of Gao and Lau,[11] the following constants are introduced[12][13]:

$$J(a, X) = \sup\Big\{ \min\{\|x+y\|, \|x-z\|\} : x, y, z \in B_X, \\ \|y-z\| \leq a\|x\| \Big\}$$

$$C_{\mathrm{NJ}}(a, X) = \sup\Big\{ \frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x, y, z \in X, \\ \|x\| + \|y\| + \|z\| \neq 0, \|y-z\| \leq a\|x\| \Big\}.$$

[11] On two classes of Banach spaces with uniform normal structure. Studia Math. 99 (1991), no. 1, 41–56.

[12] Dhompongsa, Kaewkhao, and Tasena, J. Math. Anal. Appl. 285 (2003), no. 2, 419–435.

[13] Dhompongsa, Piraisangjun, and Saejung, Bull. Austral. Math. Soc. 67 (2003), no. 2, 225–240.

$$J(a,X) = \sup\left\{ \min\{\|x+y\|, \|x-z\|\} : x,y,z \in B_X, \|y-z\| \le a\|x\| \right\}$$

$$C_{\mathrm{NJ}}(a,X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x,y,z \in X, \|x\| + \|y\| + \|z\| \neq 0, \|y-z\| \le a\|x\| \right\}.$$

Remark:

- $J(0,X) = J(X)$
- $C_{\mathrm{NJ}}(0,X) = C_{\mathrm{NJ}}(X)$

Improved sufficient conditions:[15]

The following is an improvement of [14]:

$$J(a,X) < \frac{3+a}{2} \text{ for some } 0 \le a \le 1$$
$$\Rightarrow J(a,X) < \frac{1-a+\sqrt{(1-a)^2+4(1+a)^2}}{2} \text{ for some } 0 \le a \le 1$$
$$\Rightarrow X \text{ has normal structure}$$

Remark: For all $0 \le a < 1$,

$$\frac{3+a}{2} < \frac{1-a+\sqrt{(1-a)^2+4(1+a)^2}}{2}.$$

[14] Dhompongsa, Kaewkhao, and Tasena, J. Math. Anal. Appl. 285 (2003), no. 2, 419–435.

[15] J. Gao and S. Saejung, Nonlinear Analysis Series A: Theory, Methods & Applications, 71(7-8) (2009), 3047-3052.

Remark: We also obtain an improvement for the generalized NJ-constant.

An answer of Llorens-Fuster's question[16]

Llorens-Fuster proved that

$$C_Z(X) < \frac{16}{13} \Rightarrow X \text{ has normal structure,}$$

where

$$C_Z(X) = \sup\left\{ \frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} : \|x\| + \|y\| \neq 0 \right\}.$$

He asked this question:

"*Is $16/13$ sharp in this situation?*"

Remark: $C_Z(X) \leq C_{\mathrm{NJ}}(X)$ and there is a Banach space X such that $C_Z(X) < C_{\mathrm{NJ}}(X)$.

[16] E. Llorens-Fuster, Zbăganu constant and normal structure, Fixed Point Theory 9 (2008) 159–172.

The main tool that Llorens-Fuster used in his paper is the modified Hexagonal Lemma[17]. A careful application of this lemma gives the following result:

$$C_Z(X) < \frac{16}{13} \approx 1.23$$

$$\Rightarrow C_Z(X) < \frac{1+\sqrt{3}}{2} \approx 1.36$$

X has normal structure

Remark: It seems to be unknown whether $\frac{1+\sqrt{3}}{2}$ is sharp in this situation.

[17] S. Saejung, J. Math. Anal. Appl. 330 (2007), no. 1, 597–604.

Quantitative results:

Bynum[18] defined the *weakly convergent sequence coefficient* of X by

$$\mathrm{WCS}(X) = \inf\left\{\frac{\lim_{k\to\infty}\sup\{\|x_n - x_m\| : n, m \geq k\}}{\inf\left\{\limsup_{n\to\infty}\|x_n - y\| : y \in \mathrm{co}(\{x_n\})\right\}}\right\}$$

where the infimum is taken over all weakly convergent sequences $\{x_n\}$ which are not norm convergent.
It is clear that for reflexive spaces X

$$\mathrm{WCS}(X) > 1 \Rightarrow X \text{ has normal structure.}$$

[18] Normal structure coefficients for for Banach spaces. Pacific J. Math. **86** (1980), 427-436.

Suppose that a Banach space X fails the Schur property, that is, X contains a weakly convergent sequence which is not norm convergent. Then[19]

- $\mathrm{WCS}(X) \geq \frac{1}{J(X)+\frac{1-\sqrt{5}}{2}}$.
- $\mathrm{WCS}(X) \geq \frac{1}{J(X)}\left(1+\frac{1}{\mu(X)}\right)$.
- $\mathrm{WCS}(X) \geq \frac{1}{C_{\mathrm{NJ}}(X)}\left(1+\frac{1}{\mu(X)^2}\right)$.

In particular, X has normal structure if $J(X) < \frac{1+\sqrt{5}}{2}$ or $J(X) < 1+\frac{1}{\mu(X)}$, or $C_{\mathrm{NJ}}(X) < 1+\frac{1}{\mu(X)^2}$.

[19] E. Casini, P. L. Papini, and S. Saejung, J. Math. Anal. Appl. 346 (2008) 177-182.