Weak convergence of a projection algorithm for variational inequalities with monotone operators defined on the dual space of a Banach space

Somyot Plubteing

Department of Mathematics, Faculty of Science, Naresuan University,
Phitsanulok 65000, Thailand.

Franco - Thai Seminar in Pure and Applied Mathematics
Bangkok, Thailand
October 29-31, 2009

Outline

Outline

1. Introduction
   - Abtract
   - Hilbert spaces and Banach spaces
   - Uniformly convex and uniformly smooth Banach spaces
   - Metric projection
2. Variational inequalities
   - Monotone operators
   - Variational inequalities for monotone operators on Hilbert spaces
   - Variational inequalities for monotone operators on Banach spaces
   - Variational inequalities for monotone operators defined on the dual space of a Banach space
3. An existence theorem and some properties
   - An existence theorem
   - Some properties
4. Weak convergence theorem
5. Application

Abtract

Let E be a smooth Banach space, let E^* be the dual space of E and let C be a nonempty closed subset of E such that JC is closed and convex subset of E^*, where J is the duality mapping on E. Let A be a monotone operator of JC into E. Then, the variational inequality problem is to find

$$u \in C \ \text{ such that } \ \langle AJu, Jv - Ju \rangle \geq 0, \ \ \forall\ v \in C.$$

In this talk, we first prove an existence theorem of the variational inequality problem for monotone operators defined on the dual space of E. Secondary, we prove a weak convergence theorem for finding a solution of the variational inequality problem by using projection algorithm method with a new projection which was introduce by Ibaraki and Takahashi [T. Ibaraki, W. Takahashi, A new projection and convence theorems for the projections in Banach spaces, Journal of Approximation Theory 149 (2007), 1-14]. Further, we apply our convergence theorem to the convex minimization problem and the problem of finding a zero point of maximal monotone operator.

Hilbert spaces and Banach spaces

Definition

Let E be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : E \to \mathbb{R}$ is said to be **a norm on X** if it satisfies the following conditions:

(1) $\|x\| \geq 0, \forall x \in E$;

(2) $\|x\| = 0 \Leftrightarrow x = 0$;

(3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in E$;

(4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in E$ and $\forall \alpha \in \mathbb{K}$.

Definition

Let $(E, \|\cdot\|)$ be a normed space.

A sequence $\{x_n\} \subset E$ is said to be **convergent in X** if there exists $x \in E$ such that $\lim\limits_{n\to\infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ to mean that x is the limit of the sequence $\{x_n\}$.

Hilbert spaces and Banach spaces

Definition

Let $(E, \|\cdot\|)$ be a normed space.
A sequence $\{x_n\} \subset E$ is said to be **a Cauchy sequence** if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\ m, n \geq N$.
That is, $\{x_n\}$ is **a Cauchy sequence** in B if and only if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$.

Definition

A normed space X is called **complete** if every Cauchy sequence in X converges to an element in X.

Definition

A complete normed linear space over field $\mathbb{K}$ is called **Banach space over $\mathbb{K}$**

Hilbert spaces and Banach spaces

Definition

The real-value function of two variables $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called **inner product** on a real vector spaces X if and only if it satisfies the following conditions
(1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and all real number α and β;
(2) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$; and
(3) $\langle x, x \rangle \geq 0$ for each $x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$. A **real inner product space** is a real vector spaces equipped with an inner product.

Definition

A **Hilbert space** is an inner product spaces which is complete under the norm induced by its inner product.

Uniformly convex and uniformly smooth Banach spaces

Definition

(1) A Banach space E is said to **strictly convex** if

$$\|x\| = \|y\| = 1, x \neq y \text{ imply } \frac{\|x+y\|}{2} < 1.$$

(2) A Banach space E is said to be **uniformly convex** if for all $\varepsilon \in (0,2]$ there exits $\delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1 \text{ with } \|x-y\| \geq \varepsilon \text{ implies } \frac{\|x+y\|}{2} < 1 - \delta_\varepsilon.$$

(3) The norm of E is also said to be **Frèchet differentiable** if for each x in unit sphere $U = \{x \in E : \|x\| = 1\}$ the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

is attained uniformly for $y \in U$.

Uniformly convex and uniformly smooth Banach spaces

Definition

Let E be a norm linear space with dim$E \geq 2$.
The **modulus of smoothness** of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$

The space E is said to be **smooth** if $\rho_E(\tau) > 0, \ \forall \tau > 0$. E is called **uniformly smooth** if and only if $\lim_{t\to 0^+} \frac{\rho_E(\tau)}{t} = 0$. Let $p > 1$. E is said to be $p-$**uniformly smooth** if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^p, \ t > 0$.

Uniformly convex and uniformly smooth Banach spaces

Definition

For each $p > 1$, the **generalized duality mapping** $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\} \qquad (1.1)$$

for all $x \in E$.

In particular, $J = J_2$ is called the **normalized duality mapping**. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

Uniformly convex and uniformly smooth Banach spaces

Example of $p-$uniformly smooth

$$L_p\ (l_p) \text{ or } (W_m)^p \text{ is } \begin{cases} \text{2-uniformly smooth} & \text{if } p \geq 2 \\ \text{p-uniformly smooth} & \text{if } 1 < p \leq 2. \end{cases}$$

We observe that every p-uniformly smooth Banach space is uniformly smooth. Furthermore, from the proof of [16, Remark 5, p.208], we have the following lemma

Lemma

Let E be a 2-uniformly smooth Banach space. Then, for all $x, y \in E$, there exists a constant $c > 0$ such that

$$\|Jx - Jy\| \leq c\|x - y\|, \tag{1.2}$$

where J is the normalized duality mapping of E.

Metric Projection on Hilbert space

Definition

H : Hilbert space
$C \subset H$: closed convex
For $x \in H$, there exists a unique $z \in C$ such that

$$\|x - z\| = \min\{\|x - y\| : y \in C\}$$

Putting $z = P_C x$, we call such P_C the **metric projection** onto C.
For $x \in H$ and $z \in C$,

$$z = P_C x \Longleftrightarrow \langle x - z, z - y\rangle \geq o, \ \forall y \in C;$$

Property of Metric Projection on Hilbert space

Note:

(1) P_C is a nonexpansive mapping of H onto C;

(2) P_C is a firmly nonexpansive mapping, i.e.,
$\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2$ for all $x, y \in H$;

(3) $\langle x-P_Cx, y-P_Cx\rangle \leq 0$ for all $x, y \in H$;

(4) $\|x-y\|^2 \geq \|x-P_Cx\|^2 + \|y-P_Cx\|^2$ for all $x \in H, y \in C$.

Metric projection on Banach space

Definition

E : a reflexive, strictly convex and smooth Banach space
$C \subset E$: closed and convex
For $x \in E$, there exists a unique $z \in C$ such that

$$\|x - z\| = \min\{\|x - y\| : y \in C\}$$

Putting $z = P_C x$, we call such P_C the **metric projection** onto C.
For $x \in E$ and $x_0 \in C$,

$$x_0 = P_C x \Longleftrightarrow \langle x_0 - y, J(x - x_0)\rangle \geq o, \forall y \in C;$$

Generalized metric projection

Definition

E : a smooth Banach space
Then, ϕ is the function of E into $\mathbb{R}$ define by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for all } x, y \in E,$$

where

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

In the case when E is a Hilbert space,

$$\phi(x,y) = \|x-y\|^2 \quad \text{for all } x, y \in E$$

Generalized metric projection

Theorem (Alber, Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, 1996 and Kamimura, Takahashi, SIAM J. Optim., 2002)

E : a reflexive, strictly convex and smooth Banach space
$C \subset E$: closed and convex
For each $x \in E$, there exists unique $x_0 \in C$ such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x)$$

Definition

Putting $x_0 = \Pi_C(x)$, we call such $\Pi_C(x)$ the **generalized projection** onto C

Generalized metric projection

Lemma (Alber, Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, 1996 and Kamimura, Takahashi, SIAM J. Optim., 2002)

E : *a reflexive, strictly convex and smooth Banach space*
$C \subset E$: *closed and convex*
$x \in E$, $x_0 \in C$
Then, $x_0 = \Pi_C x$ *if and only if* $\langle x_0 - y, Jx - Jx_0 \rangle \geqslant 0$ *for all* $y \in C$*.*

Sunny nonexpansive retraction

Definition

E : a Banach space
C : nonempty closed subset of E.
Then a mapping $Q : E \longrightarrow C$ is said to be **sunny** if

$$Q(Qx + t(x - Qx)) = Qx, \quad \forall t \geq 0.$$

A mapping $Q : E \longrightarrow C$ is said to be a **retraction** if $Qx = x, \ \forall x \in C$.

Definition

E : a Banach space
C : nonempty subset of E.
Then a mapping $T : C \longrightarrow E$ is said to be **nonexpansive** if

$$\|Tx - Ty\| \leq \|x - y\|, \ \ \forall \ x, \ y \in C$$

Sunny nonexpansive retraction

Theorem (Ibaraki and Takahashi, J. Approx. Theory, 2007)

E : a smooth Banach space
C : a nonempty closed subset of E
Q_C a retraction of E onto C
Then Q_C is **sunny and nonexpansive** *if and only if for any $x \in E$,*

$$\langle x - Q_C x, J(Q_C x - y)\rangle \geq 0, \ \ \forall y \in C$$

Sunny generalized nonexpansive retraction

Definition

E : a smooth Banach space
D : a nonempty closed convex subset of E
A mapping $R : D \to D$ is called **generalized nonexpansive** if $F(R) \neq \emptyset$ and

$$\phi(Rx, y) \leq \phi(x, y), \quad \forall x \in D, \forall y \in F(R)$$

where $F(R)$ is the set of fixed points of R.
R is called **sunny generalized nonexpansive retraction** if R is sunny, generalized nonexpansive and retraction.

Sunny generalized nonexpansive retraction

Lemma (Kohsaka and Takahashi, J. Nonlinear Convex Anal., 2007)

E : a smooth and strictly convex Banach space
C : a nonempty closed subset of E
Then there exists a sunny generalized nonexpansive retraction R_C of E onto C if and only if $J(C)$ is closed and convex.

In this case R_C is given by $R_C = J^{-1}\Pi_{J(C)}J$

Sunny generalized nonexpansive retraction

Theorem (Ibaraki and Takahashi, J. Approx. Theory, 2007)

E : a smooth Banach space
C : a nonempty closed subset of E
R_C a retraction of E onto C
Then R_C is **sunny and generalized nonexpansive** *if and only if*

$$\langle x - R_C x, J(R_C x) - J(y)\rangle \geq 0, \ \ \forall y \in C$$

for each $x \in E$ and $y \in C$.

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Monotone operators

Definition

H: Hilbert space
$C \subset H$: closed convex
$A : C \to H$: an operator

A is said to be **monotone** $\Leftrightarrow \langle Au - Av, u - v\rangle \geq 0, \ \forall u, v \in C.$

Definition

E: a Banach space with the dual space E^*
$C \subset E$: closed convex
$A : C \to E^*$: an operator

A is said to be **monotone** $\Leftrightarrow \langle u - v, Au - Av\rangle \geq 0, \ \forall u, v \in C.$

Monotone operators

Definition

E: a Banach space with the dual space E^*
$C^* \subset E^*$: closed convex
$A : C^* \to E$: an operator
A is said to be **monotone** $\Leftrightarrow \langle Au^* - Av^*, u^* - v^* \rangle \geq 0, \ \forall u^*, v^* \in C^*.$

Variational inequalities for monotone operators on Hilbert spaces.

Definition

H : a Hilbert space
C : a nonempty closed convex subset of H
A : a monotone operator of C into H.

The variational inequality problem is to find a point $u \in C$ such that

$$\langle Au, v-u \rangle \geq 0, \quad \text{for all } \ v \in C. \tag{2.1}$$

Such a point $u \in C$ is called a solution of the problem and the set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Variational inequalities for monotone operators on Banach spaces.

Definition

E : a Banach space with the dual space E^*
C : a nonempty closed convex subset of E
A : a monotone operator of C into E^*.

The variational inequality problem is to find a point $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0, \quad \text{for all } \ v \in C. \tag{2.2}$$

Such a point $u \in C$ is called a solution of the problem and the set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Variational inequalities for monotone operators defined on the dual space of a Banach space.

Definition

E : a smooth Banach space with the dual space E^*
C : a nonempty closed subset of E such that JC is closed and convex subset of E^*, where J is the duality mapping on E.
A : a monotone operator of JC into E.

The variational inequality problem is to find a point $u \in C$ such that

$$\langle AJu, Jv - Ju, \rangle \geq 0, \quad \text{for all } v \in C. \tag{2.3}$$

Such a point $u \in C$ is called a solution of the problem and we denoted the set of solution of the variational inequality problem (2.3) by $VI(JC, A)$.

Outline

An existence theorem

Monotone and Hemicontinuous operators

Definition

E: a Banach space with the dual space E^*
$C^* \subset E^*$: closed convex
$A : C^* \to E$: an operator
A is said to be **monotone** $\Leftrightarrow \langle Au^* - Av^*, u^* - v^* \rangle \geq 0,\ \forall u^*, v^* \in C^*$.

Definition

E: a Banach space with the dual space E^*
$C^* \subset E^*$: closed convex
$A : C^* \to E$: an operator
A is said to be **hemicontinuous** if for all $x^*, y^* \in C^*$, the mapping f of $[0, 1]$ into E defined by $f(t) = A(tx^* + (1-t)y^*)$ is continuous.

An existence theorem

Lemma (3.1)

Let E be a Banach space with the dual space E^*. Let C^* be a nonempty, compact, convex subset of E^* and A a monotone operator of C^* into E. Then there exists $x_0^* \in C^*$ such that

$$\langle Ax^*, x^* - x_0^* \rangle \geq 0, \quad \forall x^* \in C^*.$$

Lemma (3.2)

Let E be a Banach space with the dual space E^*. Let C^* be a nonempty, convex subset of E^* and A a monotone, hemicontinuous operator of C^* into E. Let $x_0^* \in C^*$. Then

$$\langle Ax_0^*, x^* - x_0^* \rangle \geq 0, \quad \forall x^* \in C^* \tag{3.1}$$

if and only if

$$\langle Ax^*, x^* - x_0^* \rangle \geq 0, \quad \forall x^* \in C^*. \tag{3.2}$$

An existence theorem

Using Lemma 3.1 and Lemma 3.2, we obtained the following Theorem

Theorem (3.3)

Let E be a Banach space with the dual space E^*. Let C^* be a nonempty, compact, convex subset of E^* and A a monotone, hemicontinuous operator of C^* into E. Then there exists $x_0^* \in C^*$ such that

$$\langle Ax_0^*, x^* - x_0^* \rangle \geq 0, \quad \forall x^* \in C^*.$$

We note from Theorem 3.5 that if JC is compact and convex and A is a monotone, hemicontinuous operator of JC into E, then $VI(JC, A)$ is nonempty.

Some properties

Lemma (3.4)

Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. Let A be a monotone operator of JC in to E. Then

$$u \in VI(JC, A) \text{ if and only if } u = R_C(u - \lambda AJu), \quad \forall\, \lambda > 0,$$

where R_C is sunny generalized nonexpansive retraction of E onto C.

Some properties

Definition

Let E be a Banach space with the dual space E^*. A monotone operator $T \subset E \times E^*$ is said to be **maximal** if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator.

We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. If T is maximal monotone, then the solution set $T^{-1}0$ is closed and convex.

Definition

Let C be a nonempty closed convex subset of a Banach space E and $x \in C$. Then the **normal cone** for C at a point $x \in C$ is defined by

$$N_C(x) = \{x^* \in E^* : \langle x - y, x^* \rangle \geq 0 \text{ for all } y \in C\},$$

and is denoted by $N_C(x)$.

Some properties

Theorem (R.T. Rockafellar, Trans. Amer. Math. Soc. 149 (1970))

Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$.

Some properties

Let E be a Banach space with the dual space E^* and let C be a nonempty closed subset of E such that JC is closed and convex. Then we denote by $N_{JC}(x^*)$ the **normal cone** for JC at a point $x^* \in JC$, that is,

$$N_{JC}(x^*) = \{x \in E : \langle x, x^* - y^* \rangle \geq 0 \text{ for all } y^* \in JC\}$$

Theorem (3.5)

Let C be a nonempty, closed subset of a smooth Banach space E such that JC is closed and convex and let A be a monotone, hemicontinuous operator of JC into E. Let $B \subset E^* \times E$ be an operator defined as follows:

$$Bv^* = \begin{cases} Av^* + N_{JC}(v^*), & v^* \in JC, \\ \emptyset, & v^* \notin JC. \end{cases}$$

Then B is maximal monotone and $(BJ)^{-1}0 = VI(JC, A)$.

Some properties

Corollary (3.6)

Let E be a reflexive, strictly convex and smooth Banach space with a Fréchet differentiable norm and let C be a nonempty closed subset of E such that JC is closed and convex and let A be a monotone, hemicontinuous operator of JC into E such that $VI(JC, A) \neq \emptyset$. Then $VI(JC, A)$ is closed and $JVI(JC, A)$ is closed and convex.

Outline

Definition

E : a real Banach space with the dual space E^*
$A : D(A) \subset E :\to E^*$: an operator.

(1) A is said to be **inverse-strongly-monotone** if there exists a positive real number α such that $\langle x - y, Ax - Ay, \rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in D(A)$. In such a case A is said to be **α-inverse-strongly-monotone**.

(2) A is said to be **Lipscitz continuous** if there exists $L \geq 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$, for all $x, y \in D(A)$.

If A is α-inverse-strongly-monotone, then A is Lipscitz continuous, that is, $\|Ax - Ay\| \leq (\frac{1}{\alpha})\|x - y\|$, for all $x, y \in D(A)$.

Theorem (Iiduka and Takahashi, J. Math. Anal. Appl. 339(2008))

E : a 2-uniformly convex and uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous.
C : a nonempty closed convex subset of E
$A : C \to E^*$: α-inverse-strongly-monotone with $VI(C,A) \neq \emptyset$ and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(C,A)$.
$\{x_n\}$: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \quad n = 1, 2, ...,$$

where Π_C is the generalized projection from E onto C, $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, where c is the 2-uniformly convex constant of E.
Then the sequence $\{x_n\}$ converges weakly to an element v in $VI(C,A)$. Further $v = \lim_{n\to\infty} \Pi_{VI(C,A)}(x_n)$.

Weak convergence theorem

Definition

E : a real Banach space with the dual space E^*
$A : D(A) \subset E^* :\to E$: an operator.

(1) A is said to be **inverse-strongly-monotone** if there exists a positive real number α such that $\langle Ax^* - Ay^*, x^* - y^* \rangle \geq \alpha \|Ax^* - Ay^*\|^2$ for all $x^*, y^* \in D(A)$. In such a case A is said to be **α-inverse-strongly-monotone**.

(2) A is said to be **Lipscitz continuous** if there exists $L \geq 0$ such that $\|Ax^* - Ay^*\| \leq L\|x^* - y^*\|$, for all $x^*, y^* \in D(A)$.

If A is α-inverse-strongly-monotone, then A is Lipscitz continuous, that is, $\|Ax^* - Ay^*\| \leq (\frac{1}{\alpha})\|x^* - y^*\|$, for all $x^*, y^* \in D(A)$.

Weak convergence theorem

Before proving our theorem we need the following Lemma.

Lemma (4.1)

Let C be a nonempty closed subset of a uniformly convex and smooth Banach space E such that JC is closed and convex. Let $\{x_n\}$ be a sequence in E such that, for all $u \in C$,

$$\phi(x_{n+1}, u) \leq \phi(x_n, u) \tag{4.1}$$

for every $n = 1, 2, \ldots$. Then $\{R_C(x_n)\}$ is a Cauchy sequence, where R_C is sunny generalized nonexpansive retraction of E onto C.

Now, we can prove the following weak convergence theorem.

Theorem (4.2)

Let E *be a uniformly convex and* 2*-uniformly smooth Banach space whose duality mapping* J *is weakly sequentially continuous. Let* C *be a nonempty closed subset of* E *such that* JC *is closed and convex and let* A *be an* α*-inverse-strongly-monotone operator of* JC *into* E *such that* $VI(JC, A) \neq \emptyset$ *and* $\|AJy\| \leq \|AJy - AJu\|$ *for all* $y \in C$ *and* $u \in VI(JC, A)$. *Let* $\{x_n\}$ *be a sequence defined by* $x_1 = x \in C$ *and*

$$x_{n+1} = R_C(x_n - \lambda_n AJx_n), \tag{4.2}$$

for every $n = 1, 2, ...,$ *where* R_C *is sunny generalized nonexpansive retraction of* E *onto* C, $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < \frac{\alpha}{c}$, *where* c *is a constant in (1.2). Then the sequence* $\{x_n\}$ *converges weakly to some element* $z \in VI(JC, A)$. *Further* $z = \lim_{n\to\infty} R_{VI(JC,A)}(x_n)$.

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Application

Definition

E : a real Banach space with the dual space E^*
f : a continuously Fréchet differentiable, convex function on E^*
$x \in E$ and $x^* \in E^*$
x^* is the **gradient of f** if $\langle x^*, y - x\rangle \leq f(y) - f(x)$ for all $y \in E$
and we denote by $x^* = \nabla f(x)$

Lemma (5.1) (J.B. Baillon and G. Haddad, Israel J. Math. 115(1977))

Let E be a Banach space, f a continuously Fréchet differentiable, convex function on E^* and ∇f the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse-strongly-monotone.

Theorem (5.2)

Let E *be a uniformly convex and* $2-$*uniformly smooth Banach space whose duality mapping* J *is weakly sequentially continuous. Let* A *be an* α*-inverse-strongly-monotone of* E^* *into* E *with* $A^{-1}0 \neq \emptyset$*. Let* $x_1 = x \in E$ *and* $\{x_n\}$ *is given by*

$$x_{n+1} = x_n - \lambda_n AJx_n,$$

for every $n = 1, 2, ...,$ *where* $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < \frac{\alpha}{c}$*, where* c *is a constant in (1.11). Then the sequence* $\{x_n\}$ *converges weakly to some element* z *in* $(AJ)^{-1}0$*. Further* $z = \lim_{n\to\infty} R_{(AJ)^{-1}0}(x_n)$*.*

From Lemma 5.1 and Theorem 5.2 we get the following corollary.

Corollary (5.3)

Let E be a uniformly convex and $2-$uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Assume that f is a function on E^* that satis?es the following conditions:

1. *f is a continuously Fréchet differentiable, convex function on E^*, ∇f is $1/\alpha$-Lipschitz continuous*
2. *$(\nabla f)^{-1}0 = \{z^* \in E^* : f(z^*) = \min_{y^* \in E^*} f(y^*)\} \neq \emptyset$.*

Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and

$$x_{n+1} = x_n - \lambda_n (\nabla f) J x_n,$$

for every $n = 1, 2, ...,$ where $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{\alpha}{c}$, where c is a constant in (1.11). Then the sequence $\{x_n\}$ converges weakly to some element z in $((\nabla f)J)^{-1}0$. Further $z = \lim_{n\to\infty} R_{((\nabla) f J)^{-1}0}(x_n)$.

Finish

Thank you

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