Stabilization of linear non-autonomous systems with norm bounded controls

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1. Introduction

The purpose of this paper is to study the stabilization problem of linear non-autonomous control systems with norm bounded controls

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \geq 0,$$
$$\|u(t)\| \leq r, \quad \forall t \geq 0.$$

1.1 Autonomous System

For autonomous systems, where the constant matrix A satisfies some appropriate spectral properties, the paper [12, M. Slemrod] proposed the nonsmooth feedback control of the form

$$u(t) = \begin{cases} -r\frac{B^T x(t)}{\|B^T x(t)\|}, & \text{if} \quad \|B^T x(t)\| \geq r, \\ -B^T x(t), & \text{if} \quad \|B^T x(t)\| \leq r. \end{cases}$$

The paper [2, H. Bounit and H. Hammouri] extended to the smooth feedback control

$$u(t) = -r\frac{B^T x(t)}{1 + \|B^T x(t)\|},$$

and showed that this feedback control stabilizes the autonoumous system under some appropriate assumptions on the contraction semigroup. It is worth noting that the approach in these works can not be readily applied to the non-autonomous systems.

1.2 Nonautonomous System

The main difficulty is that the investigation of the spectrum of the time-varying matrix/operator input $A(t)$ or its evolution matrix/operator is still complicated and there are no appropriate properties available as in the autonoumous case.

1.3 Objective

To find normed bounded control $u(t)$ for linear *<u>Nonautonomous</u>* control system.

2 Notations and preliminaries

The following notations and definitions are used throughout this paper.

R^+ is the set of all non-negative real numbers;

R^n is the Euclidean n finite-dimensional space, with the vector norm $||.||$ and the scalar product $\langle x, y\rangle$ of two vectors x, y.

$R^{n\times m}$ is the set of all $(n \times m)-$ matrices;

A^T is the transpose of the matrix A, matrix A is symmetric if $A = A^T$;
I is the identity matrix;
$\lambda(A)$ is the set of all eigenvalues of A;
$\lambda_{\max}(A) = \max\{\mathrm{Re}\lambda : \lambda \in \lambda(A)\}, \lambda_{\min}(A) = \min\{\mathrm{Re}\lambda : \lambda \in \lambda(A)\}$;
$\eta(A)$ is the matrix measure of the matrix A defined by

$$\eta(A) = \frac{1}{2}\lambda_{\max}(A + A^T);$$

$||A||$ is the spectral norm of the matrix defined by

$$||A|| = \sqrt{\lambda_{\max}(A^T A)};$$

The matrix fucntion $A(t) \in R^{n \times n}$ is bounded on R^+ if

$$\exists M > 0 : \quad \sup_{t \in R^+} \|A(t)\| \leq M;$$

$L_2([t, s], R^m)$ is the set of all measurable L_2-integrable and R^m-valued functions on $[t, s]$;

Matrix $Q \in R^{n\times n}$ is positive semidefinite $(Q \geq 0)$ if $\langle Qx, x\rangle \geq 0$, for all $x \in R^n$. If $\langle Qx, x\rangle > 0$ for all $x \neq \{0\}$, then Q is positive definite $(Q > 0)$.

$M([0,\infty), R^n_+)$ is the set of all symmetric positive semidefinite matrix functions, which are continuous and bounded on $[0,\infty)$.

Consider the system described by

$$\dot{x} = f(x, t) \tag{1.1}$$

Definition 1.1 If $f(c, t) = 0$ for all t, where c is some constant vector, then it follows at once from (1.1) that if $x(t_0) = c$ then $x(t) = c$ for all $t \geq t_0$. Thus solutions starting at c remain there, and c is said to be an *equilibrium* or *critical point*.

Definition 1.3 An equilibrium state $x = 0$ is said to be

1. **Stable** if for any positive scalar ε there exists a positive scalar δ such that $\|x(t_0)\|_e < \delta$ implies $\|x(t)\|_e < \varepsilon$, $t \geq t_0$, where $\|.\|_e$ is a standard Eucledian norm.

2. **Asymptotically stable** if it is stable and if in addition $x(t) \to 0$ as $t \to +\infty$.

2.1 Definitions

Consider linear non-autonomous control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in R^+, \tag{1}$$

where $x(t) \in R^n, u(t) \in R^m, A(t) \in R^{n\times n}, B(t) \in R^{n\times m}$ and the control $u(t) \in L_2([0, N], R^m), N > 0$ satisfies the following bounded condition

$$||u(t)|| \leq r, \quad \forall t \in R^+. \tag{2}$$

Definition 2.1. The system (1) is globally stabilizable if there is a feedback control $u(t) = k(x(t))$ satisfying the constraint (2) such that the resulting closed-loop system of (1):

$$\dot{x}(t) = A(t)x(t) + B(t)k(x(t)), \quad t \in R^+,$$

is globally asymptotically stable in the Lyapunov sense.

Definition 2.2. We recall from [6] that linear system (1) is globally controllable (GC) in finite time if there is a number $N > 0$ such that for every $x_0 \in R^n$, there is a control $u(t) \in L_2([0, N], R^m)$ satisfying

$$U(N, 0)x_0 + \int_0^N U(N, s)B(s)u(s)ds = 0,$$

where $U(t, s)$ denotes the transition matrix of the linear time-varying system $\dot{x}(t) = A(t)x(t)$ defined by

$$\frac{\partial U(t, s)}{\partial t} = A(t)U(t, s), \quad t, s \geq 0, \quad U(t, t) = I.$$

2.2 Prelimimary Results

Associated with the non-autonomous control system (1) we consider the following Riccati differential equation (RDE):

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + P(t)B(t)B^T(t)P(t) + Q(t) = 0, \quad (3)$$

where $P(t), Q(t) \in R^{n\times n}$.

Proposition 2.1. [5, M.Ikeda et al] *If the system (1) is UGC in finite time, then the following assertions hold.*
(i) There is a number $c_5 > 0$ such that

$$\int_{t_1}^{t_2} U^T(s,t_1)U(s,t_1)ds \leq c_5(t_2 - t_1)I, \quad \forall t_2 > t_1 \geq 0.$$

(ii) The RDE (3), where $Q(t) = I$, has a solution $P(t) \in M([0,\infty), R^n_+)$, which is a bounded from above and below function. Moreover, we have

$$\|P(t)\| \leq \Big[\frac{1}{c_1} + nc_5(1 + \frac{nc_2}{c_1})^2\Big], \quad \forall t \in R^+,$$

where the positive numbers c_1, c_2 are defined by Definition 2.3

Proposition 2.2. *If the control system (1) is UGC in finite time, then the RDE (3), where* $Q = \eta I$*, has a solution* $P(t) \in M([0,\infty), R_+^n)$ *satisfying*

$$\|P(t)\| \leq \Big[\frac{1}{\eta c_1} + n c_5 (1 + \frac{n c_2}{c_1})^2\Big]\eta, \quad \forall t \in R^+.$$

Proposition 2.3. *Let $B(t), P(t)$ be bounded matrix functions. Then the following functions defined by*

$$f(t,x) = -r\frac{B(t)B^T(t)P(t)x}{1+\|B^T(t)P(t)x\|}, \quad t \in R^+,$$

$$g(t,x) = -r\frac{B(t)B^T(t)[P(t)+I]x}{1+\|B^T(t)[P(t)+I]x\|}, \quad t \in R^+,$$

are global Lipschitz on R^n.

Proposition 2.4. *For any symmetric matrix* $A(t) \in R^{n\times n}$, *there exist a symmetric* $Q(t) \geq 0$ *such that* $Q(t) - A(t) \geq 0, t \in R^{+}$.

Proposition 2.5. [8] *For any real matrices* A, B :

(i) $\lambda_{\min}(AB) = \lambda_{\min}(BA)$.

Proposition 2.5. [4] (Lyapunov stability theorem) *Consider the functional differential equation* $\dot{x} = f(t, x_t), x(t) = \phi(t), t \in [-h, 0]$. *If there is a function* $V(t, x_t)$ *and positive numbers* $\lambda_i, i = 1, 2, 3$ *such that for all the solution* $x(t)$ *the following conditions hold.*

(i) $\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2, \quad t \in R^+,$

(ii) $\dot{V}(x(t)) \leq -\lambda_3 \|x(t)\|^2,$

then the zero solution is asymptotically stable.

Proposition 2.6. (Lyapunov stability theorem) *Consider the functional differential equation* $\dot{x} = f(t, x_t), x(t) = \phi(t), t \in [-h, 0]$. *If there is a function* $V(t, x_t)$ *and positive numbers* $\lambda_i, i = 1, 2, 3$ *such that for all the solution* $x(t)$ *the following conditions hold.*

(i) $\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2, \quad t \in R^+,$

(ii) $\dot{V}(x(t)) \leq -\lambda_3 \|x(t)\|^2,$

then the zero solution is asymptotically stable.

3. Stabilizability Conditions

Consider linear non-autonomous control system (1), where the matrix function $B(t)$ is bounded on R^+. Denote

$$\alpha = \frac{1}{c_1}, \quad \beta = nc_5(1 + \frac{nc_2}{c_1})^2,$$

$$\gamma = \frac{1}{b^2}, \quad b = \sup_{t \in R^+} \|B(t)\|,$$

where c_1, c_2, c_5 are defined by Proposition 2.1.

In the sequel, we need the following assumptions.

A.1. Linear control system (1) is UGC in finite time

A.2. $\gamma \geq 4\alpha\beta.$

Let $\eta > 0$ be any solution of the inequation

$$\beta^2\eta^2 + (2\alpha\beta - \gamma)\eta + \alpha^2 < 0, \quad (4)$$

and consider the following RDE:

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) + \eta I = 0. \quad (5)$$

Theorem 3.1. *Suppose that assumptions A.1, A.2 hold. Then the linear control system (1) is globally stabilizable and the stabilizing control is*

$$u(t) = -\frac{rB^T(t)P(t)x(t)}{1 + \|B^T(t)P(t)x(t)\|}, \tag{6}$$

where $P(t)$ is the solution of the RDE (5).

Proof. Assume that linear control system (1) is UGC in some finite time $T > 0$. By the Assumption A.2, the inequation (4) has a solution $\eta > 0$. Consider RDE (5) and by Proposition 2.2, this Riccati equation has a solution $P(t) \in M([0, \infty), R^n_+)$ such that

$$p = \sup_{t \in R+} \|P(t)\| \leq \left[\frac{\alpha}{\eta} + \beta\right]\eta. \qquad (7)$$

Let us consider the bounded feedback control (6). By Proposition 2.3, the function

$$f(x) = -r\frac{B(t)B^T(t)P(t)x}{1+\|B^T(t)P(t)x\|}$$

is global Lipschitz and hence the closed-loop system

$$\dot{x}(t) = A(t)x(t) + f(x(t)), \quad x(0) = x_0, \tag{8}$$

is well defined. Consider the Lyapunov function

$$V(t,x) = \langle P(t)x, x\rangle.$$

It is verified that the function $V(t,x)$ is positive definite due to the uniformly boundedness from below of $P(t)$ (Proposition 2.1). Furthermore, taking the derivative of $V(.)$ along the solution $x(t)$ of the system (8), we have

$$\begin{aligned}\dot{V}(t,x)) &= \langle \dot{P}x, x\rangle + 2\langle P\dot{x}, x\rangle \\ &= -\eta\|x(t)\|^2 + \langle PBB^T Px, x\rangle \\ &\quad -\frac{2r}{1+\|B^T Px\|}\langle PBB^T Px, x\rangle \\ &\leq -\eta\|x\|^2 + \langle PBB^T Px, x\rangle, \end{aligned} \tag{9}$$

because of

$$\frac{2r}{1+\|B^T(t)P(t)x(t)\|}\langle P(t)B(t)B^T(t)P(t)x,x\rangle \geq 0.$$

Therefore, from (9) it follows that

$$\dot{V}(t,x(t)) \leq -(\eta - p^2b^2)\|x(t)\|^2,$$

and the derivative of $V(.)$ is negative if

$$\eta > p^2b^2. \tag{10}$$

Using the condition (7), we have

$$p^2 \leq (\alpha + \beta\eta)^2,$$

and by the chosen number η from the condition (6), we can verify that

$$(\alpha + \beta\eta)^2 < \frac{\eta}{b^2},$$

such that the condition (10) holds. This completes the proof of the theorem. $\square$

Example 3.1. Consider linear control system (1) in R^2, where $r = 1$, and

$$A(t) = \begin{pmatrix} \sin 2t & 0 \\ 0 & -1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} \frac{1}{500} e^{-\cos^2 t} & 0 \\ 0 & \frac{1}{500} e^{-t} \end{pmatrix}.$$

We can find that the transition matrix is given by

$$U(t,s) = \begin{pmatrix} e^{\cos^2 s - \cos^2 t} & 0 \\ 0 & e^{-(t-s)} \end{pmatrix}.$$

If we take $N = 1$, the system is UGC with

$$c_1 = \frac{1}{250000} e^{-2}, \quad c_2 = \frac{1}{250000}, \quad c_5 = 250000(e^2 + 1).$$

Therefore, we can verify the condition A.2 with

$$\gamma = \frac{1}{2b^2} = 125000 \geq 4\alpha\beta = 8e^2(e^2 + 1)(1 + 2e^2)^2,$$

and it is verified that the number $\eta = \frac{1}{2500}$ is the solution of (4). Applying Theorem 3.1, the system is stabilizable by the bounded control $\|u(t)\| \leq 1$.

Remark 3.1. It should be noted that the global uniform controllability of the system $[A(t), B(t)]$ guarantees the existence of the bounded solution of RDE (5) and therefore, we can verify the global stabilizability without solving any RDE. However, to construct the stabilizing feedback control, we need to solve RDE (5). Various efficient methods can be used to find the solution of Riccati differential equations [1, 4, 15]. For details, see [7], where two Schur techniques are proposed for finding the solution of both algebraic and differential Riccati equations.

The following theorem gives a sufficient condition for the stabilizability without solving any RDE. Consider the linear control system (1), where $B(t) \in R^{n\times n}$ is a bounded function. For any $\alpha(t) : R^{+} \rightarrow R^{+}$, we consider the following Lyapunov equation (LE)

$$\dot{P}(t) + A^{T}(t)P(t) + P(t)A(t) - B(t)B^{T}(t) + Q(t) + \alpha(t)I = 0. \quad (11)$$

Theorem 3.2. *Assume that the LE (11), where*

$$\alpha(t) > 2\eta(A(t)) \tag{12}$$

$$Q(t) - B(t)B^T(t) \geq 0, \quad t \in R^+, \tag{13}$$

has a solution $P(t) \in M([0,\infty), R^+)$. *Then the control system (1) is globally stabilizable by the feedback control*

$$u(t) = -\frac{rB^T(t)[P(t)+I)]x(t)}{1+\|B^T(t)[P(t)+I]x(t)\|}. \tag{14}$$

Proof. Similar to Theorem 3.1, except that We need to use Proposition 2.5 (i)-(ii) to estimate some terms:

$$\begin{aligned}
\langle P(t)B(t)B^T(t)x(t), x(t)\rangle &\geq \lambda_{\min}[P(t)B(t)B^T(t)]\|x\|^2 \\
&= \lambda_{\min}[B^T(t)P(t)B(t)]\|x\|^2, \\
\langle B(t)B^T(t)P(t)x(t), x(t)\rangle &\geq \lambda_{\min}[B(t)B^T(t)P(t)]\|x\|^2 \\
&= \lambda_{\min}[B^T(t)P(t)B(t)]\|x\|^2,
\end{aligned}$$

Remark 3.2. Theorem 3.2 gives sufficient conditions for the global stabilizability of linear non-autonomous system (1) in terms of the solution of Lyapunov equations. Since this equation is linear so it is easy to find its solution. The conditions do not involve any stability property of the system matrix $A(t)$.

Remark 3.3. Note that from the proof of the Theorem 3.2 we can replace the LE (11) by the following LE

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - B(t)B^T(t) + Q(t) = 0, \tag{15}$$

where $Q(t) \geq 0$ is any symmetric matrix chosen, by Proposition 2.4, satisfying the condition

$$Q(t) > A(t) + A^T(t) + B(t)B^T(t). \tag{16}$$

Example 3.2. Consider the linear control system (1), where $\|u(t)\| \leq r = 1$, and

$$A(t) = \begin{pmatrix} 0 & e^{-t} \\ -e^{-t} & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

Note that $A(t)$ is not asymptotically stable since the solution $x(t) = (x_1(t), x_2(t))$ of the linear system $\dot{x}(t) = A(t)x(t)$:

$$\begin{cases} x_1(t) = C_1 \cos e^{-t} + C_2 \sin e^{-t}, \\ x_2(t) = C_2 \cos e^{-t} - C_1 \sin e^{-t}, \end{cases}$$

does not goes to 0 as $t \to \infty$.

On the other hand, taking

$$Q(t) = \begin{pmatrix} e^{-2t} + e^{-t} & 0 \\ 0 & e^{-3t} + e^{-t} \end{pmatrix}$$

we can verify the condition (16):

$$Q(t) - [A(t) + A^T(t)] - B(t)B^T(t) = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} + e^{-t} - e^{-2t} \end{pmatrix} > 0.$$

The solution $P(t) \geq 0$ of LE (15) is determined by

$$P(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Therefore, by Theorem 3.2 the system is globally stabilizable by the stabilizing feedback control:

$$\begin{cases} u_1(t) = -\frac{e^{-2t}x_1(t)}{1+\sqrt{e^{-2t}x_1^2(t)+e^{-3t}x_2^2(t)}}, \\ u_2(t) = -\frac{e^{-3t}x_2(t)}{1+\sqrt{e^{-2t}x_1^2(t)+e^{-3t}x_2^2(t)}}. \end{cases}$$

4. H_∞ - Control problem

Consider the following uncertain LTV system with time varying delay

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + A_1(t)x(t-h(t)) + B(t)u(t) + B_1(t)w(t), \quad t \in R^+, \quad (2.1) \\ z(t) &= C(t)x(t) + D(t)u(t), \\ x(t) &= \phi(t), t \in [-h, 0], \end{aligned}$$

where $x \in R^n$ is the state; $u \in R^m$ is the control; $w \in R^p$ is the uncertain input, $z \in R^q$ is the observation output; $A(t), A_1(t), B(t), B_1(t), C(t), D(t)$ are given matrix functions continuous and bounded on R^+. The time-delay function $h(t) \in C[-h, 0]$ satisfies the condition:

$$0 \le h(t) \le h, \quad \dot{h}(t) \le \delta < 1, \quad \forall t \in R^+.$$

Definition 2.1. Linear control system (2.1) , where $w(t) = 0$, is exponentially stabilizable if there exist a feedback control $u(t) = K(t)x(t)$, such that the zero solution of the closed-loop delay system

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t) + A_1(t)x(t - h(t), \tag{2.2}$$

is exponentially stable in the Lyapunov sense, i.e.

$$\exists N > 0, \alpha > 0 : \quad \|x(t, x_0)\| \leq N\|x_0\|e^{-\alpha t}, \quad \forall t \geq 0.$$

Definition 2.2. Given $\gamma > 0$. The H_∞ optimal control problem for the system (2.1) has a solution if there is a feedback control $u(t) = K(t)x(t)$ such that

(i) the system (2.1), where $w(t) = 0$, is exponentially stabilizable,

(ii) there is a number $c_0 > 0$ such that

$$\sup \frac{\int_0^\infty \|z(t)\|^2 dt}{c_0\|x_0\|^2 + \int_0^\infty \|w(t)\|^2 dt} \leq \gamma, \tag{2.3}$$

where the supremum is taken over all initial states x_0 and non-zero admissible uncertainties $w(t)$. In this case we say that the feedback control $u(t) = K(t)x(t)$ exponentially stabilizes the system (2.1).

Given $\gamma > 0, \mu = \frac{1}{1-\delta}$, we set

$$A_\gamma(t) = A(t) + \frac{1}{\gamma}B_1(t)B_1^T(t) + \mu A_1(t)A_1^T(t) - B(t)B^T(t),$$

$$B_\gamma(t) = [B(t)B^T(t) - \frac{1}{\gamma}B_1(t)B_1^T(t) - \mu A_1(t)A_1^T(t)]^{\frac{1}{2}}.$$

Theorem 3.1. *Assume that* $B(t)B^T(t) - \frac{1}{\gamma}B_1(t)B_1^T(t) - \mu A_1(t)A_1^T(t) \geq 0,\ t \geq 0$, *and linear control system* $[A_\gamma(t), B_\gamma(t)]$ *is globally null-controllable in finite time. Then the* H_∞ *optimal control problem for the system (2.1) has a solution. The stabilizing feedback control is defined by*

$$u(t) = -B^T(t)[P(t) + I]x(t), \quad t \in R^+,$$

where $P(t)$ *is a solution of matrix Riccati equation*

$$\dot{P} + A_\gamma^T P + PA_\gamma - PB_\gamma B_\gamma^T P + Q = 0.$$

Example 3.1.

Consider system (2.1), where $h(t) = 0.25 \sin^2 t$, and

$$A(t) = \begin{bmatrix} -\frac{3}{2} - \frac{1}{2}e^{-2\sin t} & 1 \\ -1 & -\frac{7}{4} - \frac{1}{2}e^{-2\sin t} \end{bmatrix}, \; A_1(t) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}, \; B(t) = B_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C(t) = \begin{bmatrix} \frac{1}{2}e^{-\sin t} & -\frac{1}{2}e^{-\sin t} \\ -\frac{1}{2}e^{-\sin t} & \frac{1}{2}e^{-\sin t} \end{bmatrix}, \; D = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

$$\begin{aligned} u(t) &= -B^T(t)[P(t) + I]x(t) \\ &= -\begin{bmatrix} 2 & 0 \end{bmatrix} x(t) \\ &= -2x_1(t) \end{aligned}$$

5. H_∞ - Control problem with normed bounded control

Consider the following uncertain LTV system with time varying delay

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + A_1(t)x(t-h(t)) + B(t)u(t) + B_1(t)w(t), \quad t \in R^+ \qquad (2.1)\\
z(t) &= C(t)x(t) + D(t)u(t),\\
x(t) &= \phi(t), t \in [-h, 0],
\end{aligned}$$

Further Investigations:

1. Consider the problem with normed bounded control.

2. Remove restriction $\dot{h}(t) \leq \delta < 1$**.**