

Strong convergence theorems of modified Ishikawa iterations for countable hemi-relatively nonexpansive mappings in a Banach space

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Research Article

Strong Convergence Theorems of Modified Ishikawa Iterations for Countable Hemi-Relatively Nonexpansive Mappings in a Banach Space

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Abstract

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Abstract

In this paper, we prove some strong convergence theorems for fixed points of modified Ishikawa and Halpern iterative processes for a countable family of hemi-relatively nonexpansive mappings in a uniformly convex and uniformly smooth Banach space by using the hybrid projection methods. Moreover, we also apply our results to a class of relatively nonexpansive mappings, and hence, we immediately obtain the results announced by Qin and Su's result (2007), Nilsrakoo and Saejung's result (2008), Su et al.'s result (2008) and some known corresponding results in the literatures.

Introduction

- ▶ Let E be a real Banach space, C be a nonempty closed convex subset of E , and $T : C \rightarrow C$ be a mapping. Recall that T is *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T , that is $F(T) = \{x \in C : x = Tx\}$.

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We denote by $F(T)$ the set of fixed points of T , that is $F(T) = \{x \in C : x = Tx\}$.

- ▶ A mapping T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\| \quad \text{for all } x \in C \text{ and } y \in F(T).$$

Introduction

- ▶ The Mann's iterative algorithm was introduced by Mann [10] in 1953. This iteration process is now known as *Mann's iteration process*, which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \quad (1.1)$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0, 1]$.

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where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0, 1]$.

- ▶ In 1967, *Halpern* [6] first introduced the following iteration scheme:

$$\begin{cases} x_0 = x \in C & \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n. \end{cases} \quad (1.2)$$

Introduction

- ▶ In 1974, *Ishikawa* [7] introduced a new iteration scheme, which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases} \quad (1.3)$$

where the initial guess x_0 is taken in C arbitrarily and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in the interval $[0, 1]$.

Introduction

Zhang and Su [20] introduced the following implicit hybrid method for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in a real Hilbert space:

$$\left\{ \begin{array}{l} x_0 \in C \text{ is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) T_n z_n, \\ z_n = \beta_n y_n + (1 - \beta_n) T_n y_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n = 0, 1, 2, \dots, \end{array} \right. \quad (1.4)$$

where $T_n \equiv T_{n \bmod N}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$ and $\{\beta_n\} \subset [b, 1]$ for some $b \in (0, 1]$.

Introduction

In 2008, Nakprasit et al. [13] established weak and strong convergence theorems for finding common fixed points of a countable family of nonexpansive mappings in a real Hilbert space. In the same year, Cho et al. [5] introduced the normal Mann's iterative process and proved some strong convergence theorems for a finite family nonexpansive mapping in the framework Banach spaces.

Introduction

To find a common fixed point of a family of nonexpansive mappings, Aoyama et al. [2] introduced the following iterative sequence: Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n x_n, \quad (1.5)$$

for all $n \in \mathbb{N}$, where C is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence of $[0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings. Then they proved that, under some suitable conditions, the sequence $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of $\{T_n\}$.

Introduction

In 2008, by using a (new) hybrid method, Takahashi et al. [19] proved the following theorem.

Introduction

Theorem 1.1 (Takahashi et al. [19]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let $\{T_n\}$ and \mathcal{T} be families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) := F(\mathcal{T}) \neq \emptyset$ and let $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with \mathcal{T} . For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \quad (1.6)$$

where $0 \leq \alpha_n < 1$ for all $n \in \mathbb{N}$ and $\{T_n\}$ is said to satisfy the NST-condition (I) with \mathcal{T} if for each bounded sequence $\{z_n\} \subset C$, $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$ implies that $\lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0$ for all $T \in \mathcal{T}$. Then, $\{x_n\}$ converges strongly to $P_{F(\mathcal{T})} x_0$.

Introduction

On the other hand, Matsushita and Takahashi [11] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \quad n = 0, 1, 2, \dots, \quad (1.7)$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E . Under some suitable conditions, they proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T .

Introduction

Recently, Kohsaka and Takahashi [9] extended iteration (1.7) to obtain a weak convergence theorem for common fixed points of a finite family of relatively nonexpansive mappings $\{T_i\}_{i=1}^m$ by the following iteration:

$$x_{n+1} = \Pi_C J^{-1} \left(\sum_{i=1}^m w_{n,i} (\alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) JT_i x_n) \right), \quad n = 0, 1, 2, \dots, \quad (1.8)$$

where $\alpha_{n,i} \in [0, 1]$ and $w_{n,i} \in [0, 1]$ with $\sum_{i=1}^m w_{n,i} = 1$, for all $n \in \mathbb{N}$.

Introduction

Moreover, Matsushita and Takahashi [12] proposed the following modification of iteration (1.7) in a Banach space E :

$$\left\{ \begin{array}{l} x_0 = x \in C, \quad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \quad n = 0, 1, 2, \dots, \end{array} \right. \quad (1.9)$$

and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}(x)$.

Introduction

Qin and Su [17] showed that the the sequence $\{x_n\}$, which is generated by a relatively nonexpansive mappings T in a Banach space E , as following

$$\left\{ \begin{array}{l} x_0 \in C, \quad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JTz_n), \\ C_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{array} \right. \quad (1.10)$$

converges strongly to $\Pi_{F(T)}x_0$.

Introduction

Moreover, they also showed that the the sequence $\{x_n\}$, which is generated by

$$\left\{ \begin{array}{l} x_0 \in C, \quad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT x_n), \\ C_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1 - \alpha_n)\phi(v, x_n), \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{array} \right. \quad (1.11)$$

converges strongly to $\Pi_{F(T)} x_0$.

Introduction

In 2008, Nilsrakoo and Saejung [14] used the following Mann's iterative process:

$$\left\{ \begin{array}{l} x_0 \in C, \text{ is arbitrary,} \\ C_{-1} = Q_{-1} = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_n x_n), \\ C_n = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots \end{array} \right. \quad (1.12)$$

and showed that the sequence $\{x_n\}$ converges strongly to a common fixed point of a countable family of relatively nonexpansive mappings.

Introduction

Recently, Su et al. [18] extended the results of Qin and Su [17], Matsushita and Takahashi [12] to a class of closed hemi-relatively nonexpansive mapping. Note that, since the hybrid iterative methods presented by Qin and Su [17] and Matsushita and Takahashi [12] can not be used for hemi-relatively nonexpansive mappings. Thus, as we known, Su et al. [18] showed their results by using the method as a monotone (CQ) hybrid method.

Introduction

In this paper, motivated by Qin and Su [17], Nilsrakoo and Saejung [14], we consider the modified Ishikawa iterative (1.10) and Halpern iterative processes (1.11), which is different from those of (1.10)-(1.12), for countable hemi-relatively nonexpansive mappings. By using the shrinking projection method, some strong convergence theorems in a uniformly convex and uniformly smooth Banach space are provided. Our results extend and improve the recent results by Nilsrakoo and Saejung's result [14], Qin and Su [17], Su et al. [18] and Takahashi et al.'s theorem [19] and many others.

Preliminaries

- ▶ Let E be a real Banach space with dual E^* . Denote by $\langle \cdot, \cdot \rangle$ the inner product. *The normalized duality mapping* J from E to E^* is defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad (2.1)$$

for all $x \in E$. If C is a nonempty closed convex subset of real Hilbert space H and $P_C : H \rightarrow C$ is the *metric projection*, then P_C is nonexpansive.

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for all $x \in E$. If C is a nonempty closed convex subset of real Hilbert space H and $P_C : H \rightarrow C$ is the *metric projection*, then P_C is nonexpansive.

- ▶ The function $\phi : E \times E \rightarrow \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for all } x, y \in E. \quad (2.2)$$

Preliminaries

- ▶ **Lemma 2.1** (*Kohsaka and Takahashi [9]*). Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists continuous, strictly increasing, and convex function $g : [0, 2r] \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

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$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

- ▶ Let C be a closed convex subset of E , and let T be a mapping from C into itself. The set of fixed points of T is denoted by $F(T)$. A mapping T is said to be *hemi-relatively nonexpansive* if

$$\phi(p, Tx) \leq \phi(p, x) \quad \text{for all } x \in C \quad \text{and } p \in F(T).$$

Preliminaries

- ▶ A point p in C is said to be an asymptotic fixed point of T [4] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n \rightarrow \infty} (x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$.

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- ▶ A *hemi-relatively nonexpansive* mapping T from C into itself is called relatively nonexpansive if $\hat{F}(T) = F(T)$.

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- ▶ A *hemi-relatively nonexpansive* mapping T from C into itself is called relatively nonexpansive if $\hat{F}(T) = F(T)$.
- ▶ We say that the mapping T is *relatively nonexpansive* if the following conditions are satisfied:
 - (R1) $F(T) \neq \emptyset$;
 - (R2) $\phi(p, Tx) \leq \phi(p, x)$ for each $x \in C, p \in F(T)$;
 - (R3) $F(T) = \hat{F}(T)$.

Preliminaries

- ▶ **Lemma 2.2** (*Kamimura and Takahashi* [8]). Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E . If $\phi(x_n, y_n) \rightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \rightarrow 0$.

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- ▶ **Lemma 2.3** (*Alber* [1]). Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C. \quad (2.3)$$

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$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C. \quad (2.3)$$

- ▶ **Lemma 2.4** (*Alber* [1]). Let E be a reflexive, strict convex, and a smooth real Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \quad (2.4)$$

Preliminaries

- ▶ **Lemma 2.5** (*Matsushita and Takahashi* [12]). Let E be a strictly convex and a smooth real Banach space, let C be a closed convex subset of E , and let T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.

Preliminaries

- ▶ Let C be a subset of Banach space E and let $\{T_n\}$ be a family of mappings from C into E . For a subset B of C , we say that

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- ▶ (i) $(\{T_n\}, B)$ satisfies condition *AKTT* if

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in B\} < \infty;$$

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- ▶ (i) $(\{T_n\}, B)$ satisfies condition *AKTT* if

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in B\} < \infty;$$

- ▶ (ii) $(\{T_n\}, B)$ satisfies condition **AKTT* if

$$\sum_{n=1}^{\infty} \sup\{\|JT_{n+1}z - JT_nz\| : z \in B\} < \infty;$$

Aoyama et al. [2] prove the following result which is very useful for our main result.

Preliminaries

- ▶ **Lemma 2.6** (*Aoyama et al. [2]*). Let C be a nonempty subset of Banach space E and let $\{T_n\}$ be a sequence of mappings from C into E . Let B be a subset of C with $(\{T_n\}, B)$ satisfies condition AKTT, then there exists a mapping $\tilde{T} : B \rightarrow E$ such that

$$\tilde{T}x = \lim_{n \rightarrow \infty} T_n x \quad \text{for all } x \in B$$

and $\limsup_{n \rightarrow \infty} \{\| \tilde{T}z - T_n z \| : z \in B\} = 0$.

Preliminaries

- **Lemma 2.7** (*Nilsrakoo and Saejung [14]*). Let E be a reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty subset of Banach space E and let $\{T_n\}$ be a sequence of mappings from C into E . Let B be a subset of C with $(\{T_n\}, B)$ satisfies condition *AKTT, then there exists a mapping $\hat{T} : B \rightarrow E$ such that

$$\hat{T}x = \lim_{n \rightarrow \infty} T_n x \quad \text{for all } x \in B$$

and $\limsup_{n \rightarrow \infty} \{\|J\hat{T}z - JT_n z\| : z \in B\} = 0$.

Preliminaries

- ▶ **Lemma 2.8** (*Nilsrakoo and Saejung [14]*). Let E be a reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty subset of Banach space E and let $\{T_n\}$ be a sequence of mappings from C into E . Suppose that for each bounded subset B of C , the ordered pair $(\{T_n\}, B)$ satisfies condition AKTT or condition *AKTT. Then there exists a mapping $T : B \rightarrow E$ such that

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \text{for all } x \in C$$

Main results

In this section, we establish the strong convergence theorems for finding common fixed points of a countable family of hemi-relatively nonexpansive mappings in a uniformly convex and uniformly smooth Banach space. It is worth mentioning that our main theorem generalizes recent theorems by Su et al. [18] from relatively nonexpansive mappings to a more general concept. Moreover, our results also improve and extend the corresponding results of Nilsrakoo and Saejung [14]. In order to prove the main result, we recall a concept as following: an operator T in a Banach space is closed if $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$.

Main results

Theorem 3.1 Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty bounded closed convex subset of E . Let $\{T_n\}$ be a sequence of hemi-relatively nonexpansive mappings from C into itself such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\lim_{n \rightarrow \infty} \beta_n = 1$, and let a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily and } C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_n z_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JT_n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \quad (3.1)$$

for $n \in \mathbb{N} \cup \{0\}$,

Main results

where J is the single-valued duality mapping on E . Suppose that for each bounded subset B of C , the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. Let T be the mapping from C into itself defined by $Tv = \lim_{n \rightarrow \infty} T_n v$ for all $v \in C$ and suppose that T is closed and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. If T_n is uniformly continuous for all $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Halpern iterative scheme

In this section, we prove the strong convergence theorems for finding common fixed points of a countable family of hemi-relatively nonexpansive mappings, which can be viewed as a generalization of the recently result of [17, Theorem 2.2].

Halpern iterative scheme

Theorem 4.1 Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty bounded closed convex subset of E . Let $\{T_n\}$ be a sequence of hemi-relatively nonexpansive mappings from C into itself such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0,1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and let a sequence $\{x_n\}$ in C be defined by the following algorithm:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily and } C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT_n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1 - \alpha_n)\phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \quad (4.1)$$

for $n \in \mathbb{N} \cup \{0\}$,

Halpern iterative scheme

where J is the single-valued duality mapping on E . Suppose that for each bounded subset B of C , the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. Let T be the mapping from C into itself defined by $Tv = \lim_{n \rightarrow \infty} T_n v$ for all $v \in C$ and suppose that T is closed and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.

Some Applications to Hilbert spaces

It is well known that, in the Hilbert space setting, the concepts of hemi-relatively nonexpansive mappings and quasi-nonexpansive mappings are the equivalent. Thus, the following results can be obtained.

Some Applications to Hilbert spaces

Theorem 5.1 Let H be a Hilbert space, and let C be a nonempty bounded closed convex subset of H . Let $\{T_n\}$ be a sequence of quasi-nonexpansive mappings from C into itself such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\lim_{n \rightarrow \infty} \beta_n = 1$, and let a sequence $\{x_n\}$ in C be defined by the following algorithm:

$$\left\{ \begin{array}{l} x_0 \in C, \text{ chosen arbitrarily and } C_0 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T_n z_n, \\ z_n = \beta_n x_n + (1 - \beta_n) T_n x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{array} \right. \quad (5.1)$$

for $n \in \mathbb{N} \cup \{0\}$.




Some Applications to Hilbert spaces





Suppose that for each bounded subset B of C , the ordered pair $(\{T_n\}, B)$ satisfies condition AKTT. Let T be the mapping from C into itself defined by $Tv = \lim_{n \rightarrow \infty} T_n v$ for all $v \in C$ and suppose that T is closed and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. If T_n is uniformly continuous for all $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.





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THANK YOU
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