WINDING NUMBERS
and SUMMATION PROCESSES

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Two difficult elementary problems

A short history

Winding numbers

Summation processes

Two difficult elementary problems

Pb. I $\quad (a_n)_{n \in \mathbb{Z}}$ complex

$$\sum |n| \, |a_n|^2 < \infty$$

$$\sum_m a_m \bar{a}_{n+m} = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Prove that

$$\sum n \, |a_n|^2 \in \mathbb{Z}$$

Pb. II $\quad (u_m)_{m=1,2,\dots}$ real

$$\sum \frac{|u_m|}{m} < \infty$$

$$S(t) = \sum u_m \frac{\sin mt}{mt}$$

$$S_n = u_1 + \dots + u_n$$

A) Find (u_m) such that

$\lim_{n\to\infty} S_n$ exists, $\lim_{t\to 0} S(t)$ doesn't

B) Find (u_m) such that

$\lim_{t\to 0} S(t)$ exists, $\lim_{n\to\infty} \frac{1}{n}(S_1 + \dots + S_n)$ doesn't

The mathematical context, a short history

1995

Brézis - Nirenberg Degree theory and BMO

$f \in C(S^n, S^n)$ $\qquad \deg f \in \mathbb{Z}$

$\deg f = \deg g \iff f$ homotopic to g

When $n = 1$, writing S^1 in the form $x_1 = \cos\theta$, $x_2 = \sin\theta$,
the mapping $f(x_1, x_2) = (\cos k\theta, \sin k\theta)$ has degree k
($k \in \mathbb{Z}$, the "winding number").

When $n > 1$, writing S^n in the form $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3^2 + \dots + x_{n+1}^2 = 1 - r^2$,
the mapping $f(x_1, x_2, x_3, \dots x_{n+1}) = (r\cos k\theta, r\sin k\theta, x_3, \dots, x_{n+1})$
has degree k $(k \in \mathbb{Z})$

$f \in BMO(S^n, S^n)$: bounded mean oscillation

$$\frac{1}{(\sigma(B))^2} \iint_{B\times B} \| f(x) - f(y) \| \, d\sigma(x)\, d\sigma(y) = O(1)$$

$VMO(S^n, S^n)$: vanishing mean oscillation

$$\dots\dots\dots\dots = \underset{\sigma(B)\to 0}{o(1)}$$

$f \in VMO(S^n, S^n) \to$ VMO-degree $f \in \mathbb{Z}$ (Brézis - Nirenberg)

Sobolev classes

$$W^{1,p} = \{ f \in L^p,\ \nabla f \in L^p \}$$

$$W^{s,p} = \left\{ f \in L^p;\ \iint \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy < \infty \right\}$$

$0 < s < 1$, $p > 1$; n : dimension of the domain

$$H^s = W^{s,2} \quad (0 < s < 1)$$

If $sp = n$, $W^{s,p} \subset VMO$

Examples: $H^1(S^2, S^2)$, $H^{1/2}(S^1, S^1)$

October 1995

Brezis at Rutgers

Gelfand seminar

G. What in dimension 1 ?

B. $H^{1/2}(S^1, S^1)$ $\iint \frac{|f(x)-f(y)|^2}{|x-y|^2}\,dx\,dy < \infty$

G. $e^{it} \to f(e^{it}) = \sum_{-\infty}^{\infty} a_n e^{int}$

How to express $H^{1/2}(S^1, S^1)$ on the a_n ?

B. $\sum_{-\infty}^{\infty} |n|\,|a_n|^2 < \infty$

G. How to express $\deg f$ with the a_n ?

B. ?

Comes home

$\deg f = \sum_{-\infty}^{\infty} n\,|a_n|^2$

(Solution to Problem I)

Questions on $f \in C(S^1, S^1)$ when $f \notin H^{1/2}(S^1, S^1)$

$$f(e^{it}) \sim \sum_{-\infty}^{\infty} a_n e^{int} \qquad |f(e^{it})| = 1$$

$$\deg f = \text{winding number} \qquad \sum_{-\infty}^{\infty} n|a_n|^2 \,?$$

I Summation process?

Is it possible to obtain $\deg f$ through a convenient process of summation, applied to $\sum_{-\infty}^{\infty} n|a_n|^2$?

Korevaar 1999

Yes if $f \in BV$: $\deg f = \lim_{N\to\infty} \sum_{-N}^{N} n|a_n|^2$

But $\exists f \in C$ for which partial sums $\sum_{-N}^{N} n|a_n|^2$ diverge, or converge to a value $\neq \deg f$, and the same for the Abel-Poisson sums $\sum_{-\infty}^{\infty} n r^{|n|}|a_n|^2$ $(r \uparrow 1)$

Kahane 2005

II Is $\deg f$ well defined by the $|a_n|$? "Can one hear the degree of continuous maps"? (Brézis 2006)

No, "one cannot hear the winding number",

Bourgain-Kozma 2007 (difficult paper)

III Is it true that

$$\sum_{-\infty}^{\infty} |n|\,|a_n|^2 \leq |\deg f| + \sum_{0}^{\infty} n|a_n|^2 \quad \text{(Brézis 2008)}$$

Yes, and moreover

$$\sum_{0}^{\infty} n^{2s}|a_n|^2 < \infty \Rightarrow \sum_{-\infty}^{\infty} |n|^{2s}|a_n|^2 < \infty \qquad (0 < s < \infty)$$

Bourgain-Kahane 2009

When and how can we hear the winding number?

$f(e^{it}) = e^{i\nu t} e^{i\varphi(t)}$

$\nu = \deg f \qquad \varphi$ periodic

$$\sum_{-\infty}^{\infty} |a_n|^2 e^{int} = \int_0^{2\pi} f(e^{i(t+s)}) \overline{f(e^{is})} \frac{ds}{2\pi}$$

If $\nu = 0$, $\quad \sum |a_n|^2 \sin nt = \int_0^{2\pi} \sin(\varphi(t+s) - \varphi(s)) \frac{ds}{2\pi}$

$$\vdots$$

$$\lim_{t\to 0} \sum |a_n|^2 \frac{\sin nt}{t} = 0$$

under the assumption

$$\int_0^{2\pi} |f(e^{i(t+s)}) - f(e^{is})|^3 = o(t) \quad (t \to 0)$$

$$\boxed{\begin{array}{c} f \in \lambda^3_{1/3} \quad \text{(Zygmund's notation)} \\ \Downarrow \\ \nu = \lim\limits_{t\to 0} \sum\limits_{-\infty}^{\infty} |a_n|^2 \dfrac{\sin nt}{t} \end{array}}$$

$$\lambda_{1/3} \subset \lambda^3_{1/3} \subset \Lambda_{1/3}$$

$$\omega_f(t) = o(t^{1/3}) \qquad \omega_f(t) = O(t^{1/3}) \quad (t \to 0)$$

One can write $W^3_{1/3}$ instead of $\lambda^3_{1/3}$ (Brézis 2006)

One can't write $\Lambda_{1/3}$:

$$\boxed{\begin{array}{l} \text{Given } \lambda \in \mathbb{R},\ \lambda \neq 0,\ \exists f \in \Lambda_{1/3} \text{ such that} \\ \deg f = 0 \quad \text{and} \quad \lim\limits_{t\to 0} \sum\limits_{-\infty}^{\infty} |a_n|^2 \dfrac{\sin nt}{t} = \lambda \\ \\ \exists f \in \Lambda_{1/3} \text{ such that} \\ \varliminf\limits_{t\to 0} \sum\limits_{-\infty}^{\infty} |a_n|^2 \dfrac{\sin nt}{t} = -\infty \quad \text{and} \quad \varlimsup\limits_{t\to 0} = +\infty \end{array}}$$

SUMMATION PROCESSES

Let us write $u_m = m\left(|a_m|^2 - |a_{-m}|^2\right) \quad (m = 1, 2, \dots)$

hence $\displaystyle\sum_1^\infty \frac{|u_m|}{m} < \infty.$

We already met

$$S_n = \sum_1^n u_m \qquad (n \to \infty)$$
$$A(r) = \sum_1^\infty r^m u_m \qquad (r \uparrow 1)$$
$$S(t) = \sum_1^\infty u_m \frac{\sin mt}{mt} \qquad (t \to 0)$$

A general summation process is

$$(u_m) \longrightarrow \sum_1^\infty A_m(\varepsilon)\, u_m \qquad (\varepsilon \to 0)$$
$$\lim_{\varepsilon \to 0} A_m(\varepsilon) = 1$$

Examples:

$\varepsilon = \frac{1}{n}$	$1_{[0,1]}(m\varepsilon)$	CV : convergence	
$\varepsilon = \log\frac{1}{r}$	r^m	Abel-Poisson	
$\varepsilon = t$	$\left(\frac{\sin mt}{mt}\right)^2$	Riemann	
	$\left(\frac{\sin mt}{mt}\right)^k$	(R, k)	
$\varepsilon = \frac{1}{n}$	$(1 - m\varepsilon)^+$	$(C, 1)$	Cesaro 1

$$\sigma_n = \frac{S_0 + \dots + S_{n-1}}{n} \qquad (C, 1)$$
$$\sigma_n^{(2)} = \frac{\sigma_0 + \dots + \sigma_{n-1}}{n} \qquad (C, 2) \quad \dots$$

Order:

$\mathcal{P} \to \mathcal{P}'$: $\mathcal{P}'$ "stronger than $\mathcal{P}$"

Stronger than CV : "regular"

All above are regular, except $(R, 1)$

Reference: Hardy, Divergent series (1949, 1991)

More on (R,k)

$(R,2)$ in Riemann's thesis on trigonometric series
(1854, published in 1867)

$$\begin{array}{l} (R,1) \to (C,1+\delta) \to AP \quad \text{Zygmund 1928} \\ \downarrow \\ (R,2) \to (C,2+\delta) \to AP \quad \text{Kuttner 1935} \end{array}$$

Examples show

$$\left.\boxed{\begin{array}{l} CV \not\to (R,1) \\ (R,1) \not\to (C,1) \end{array}}\right\} \text{ This is Problem II} \quad (2009)$$

question:
$(R,2) \not\to (C,2)$?