Complex Variables and Elliptic Equations
Vol. 00, No. 00, Month 200x, 1–13

RESEARCH ARTICLE

Winding numbers and summation processes

Jean–Pierre Kahane, Laboratoire de Mathématiques, Université Paris–Sud,
91405 Orsay Cedex, France
jean-pierre.kahane@math.u-psud.fr
(*Received 00 Month 200x; in final form 00 Month 200x*)

à Gérard Bourdaud, en toute amitié

Answering a question asked by I.M. Gelfand, H. Brezis gave a simple formula for the winding number of a mapping of the circle into itself in terms of the absolute values of its Fourier coefficients. This formula applies only to good classes of mappings and may fail otherwise. It has the form of a series and this series may need a convenient process of summation. The article contains optimal results about classes of functions and a special summation process. This summation process is not classical and the last part of the article explains its relation with classical summation processes.

Keywords: Winding numbers, topological degree, Fourier series, summation processes.

AMS Subject Classification: 40D25, 40G99, 42A16, 54C35, 55M25.

The continuous maps from the circle S^1 into itself, $f \in C(S^1, S^1)$, have a topological degree, or winding number,

$$\deg f = \frac{1}{2\pi i} \int_{S^1} \frac{f'(z)}{f(z)} dz\,. \tag{1}$$

Considered as unimodular continuous functions on $\mathbb{R}/2\pi\mathbb{Z}$, they also have a Fourier series:

$$f(e^{it}) \sim \sum_{-\infty}^{\infty} a_n e^{int} \tag{2}$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt\,.$$

How to express the winding numbers by means of the Fourier coefficients ? This simple question is not so old and it has already an interesting history [5] [9] [3] [4] [8] [1] [2]: it is the matter of part I. Part II is based on my note [8]: it gives the presently best process in order to compute the winding number $\deg f$ from the coefficients a_n. Part III shows the relation between this process and other summation processes.

ISSN: 0278-1077 print/ISSN 1563-5066 online

DOI: 10.1080/0278107YYxxxxxxxx
http://www.informaworld.com

1. A sketch of the history

The beginning of the story is exposed by Ham Brézis in [3], pp. 143 and sq. It is mentioned already in the basic article of Brézis and Nirenberg [5] p. 224. Brézis and Nirenberg extended the notion of topological degree to functions of vanishing mean oscillation, $f \in VMO$. Let me recall that $VMO(\Omega)$, Ω being an open set in $\mathbb{R}^n$, is the class of all functions, defined up to an additive constant, which are locally integrable on Ω and whose oscillation on any ball contained in Ω tends to 0 when the diameter of the ball tends to 0. $VMO(S^1)$ is defined in the same way, balls becoming intervals. Using this extension, Brézis and Nirenberg could define $\deg f$ when $f \in H^{1/2}(S^1, S^1)$, meaning

$$\sum_{-\infty}^{\infty} |n|\,|a_n|^2 < \infty \tag{3}$$

and recover the extension already given by L. Boutet de Monvel and O. Gabber in this case. In 1996, Brézis presented this theory in Gelfand's seminar, and then Gelfand asked the question: how to express $\deg f$ by means of the a_n ?

When (3) holds the answer is easy and clear:

$$\deg f = \sum_{-\infty}^{\infty} n|a_n|^2 . \tag{4}$$

What happens when (3) does not held ? Is it possible to give (4) a new meaning through a convenient summation process ? This question is asked by Brézis and Nirenberg as open problem [5, p. 225] and a large part of Brézis's article [3] is devoted to it.

The first answers were given by J. Korevaar in 1999 [9]. First, the symmetric partial sums converge to $\deg f$ when $f \in C \cap BV(S^1, S^1)$, BV meaning the class of functions with bounded variation. Then, more important, a negative statement: the most usual summation processes namely, symmetric partial sums and Abel–Poisson, fail for some $f \in C(S^1, S^1)$; actually, when applied to the series in (4), they can either diverge or converge to a value different from $\deg f$. Hence a second open problem stated by Brézis as "Can we hear the degree of continuous maps ?": is $\deg f$ well defined when the sequence $\{|a_n|\}_{n\in\mathbb{Z}}$ is given, assuming only $f \in C(S^1, S^1)$? [3, p. 145].

The answer is not easy. It is due to Jean Bourgain and Gady Kozma and it is negative [2, 2007].

Then Brézis turned to another question [4]. Is it true that whatever $f \in C(S^1, S^1)$

$$\sum_{-\infty}^{\infty} |n|\,|a_n|^2 \le |\deg f| + \sum_{0}^{\infty} n|a_n|^2 \tag{5}$$

holds ? Since it holds when the first member is finite, it is the same to ask whether the implication

$$\sum_{0}^{\infty} n|a_n|^2 < \infty \Longrightarrow \sum_{-\infty}^{\infty} |n|\,|a_n|^2 < \infty \tag{6}$$

is valid. Here the answer is positive and moreover

$$\sum_{0}^{\infty} n^{2s}|a_n|^2 < \infty \Longrightarrow \sum_{-\infty}^{\infty} |n|^{2s}\,|a_n|^2 < \infty \tag{7}$$

when $0 < s < 1$ (Bourgain with me, 2009, [1]). This result extends to $f \in VMO(S^1, S^1)$ but not to $f \in L^\infty(S^1, S^1)$.

My own contribution was given in 2005 in the note [8]: through a convenient process of summation (4) is valid for all functions f belonging to the Zygmund class $\lambda^3_{1/3}$ ([11] p. 45), in particular for all functions of satisfying a Hölder condition of order $> \dfrac{1}{3}$, but it fails for some $f \in \Lambda_{1/3}$, meaning hölderian of order $\dfrac{1}{3}$. The positive result was extended by Brézis to the functions $f \in W^{1/3,3}(S^1, S^1)$, that is

$$\int_0^{2\pi} \frac{|f(e^{i(t+s)}) - f(e^{is})|^3}{|e^{is}-1|^2}\,ds < \infty$$

([3], theorem 5' p. 148).

I shall describe and comment the content of the note in part II and return to summation processes in part III.

2. A way to compute the winding number

Changing $f(z)$ into $zf(z)$ is the same as translating $\{a_n\}$ by 1, and that adds 1 to both members of (4), whatever the summation process applied to the second member. Up to the final result, we therefore assume $\deg f = 0$. Then

$$f(e^{it}) = e^{i\varphi(t)}\,, \quad \varphi \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})\,, \tag{8}$$

and the sequence $\{|a_n|^2\}_{n\in\mathbb{Z}}$ is well defined by the function

$$\begin{aligned}\sum_{-\infty}^{\infty} |a_n|^2 e^{int} &= \int_0^{2\pi} f(e^{i(t+s)})\overline{f(e^{is})}\frac{ds}{2\pi} \qquad (9)\\ &= \int_0^{2\pi} e^{i(\varphi(t+s)-\varphi(s))}\frac{ds}{2\pi}\,.\end{aligned}$$

Our key formula is

$$\begin{aligned}\sum_{-\infty}^{\infty} |a_n|^2 \sin nt &= \int_0^{2\pi} \sin(\varphi(t+s)-\varphi(s))\frac{ds}{2\pi} \qquad (10)\\ &= \int_0^{2\pi} (\sin(\varphi(t+s)-\varphi(s)) - (\varphi(t+s)-\varphi(s)))\frac{ds}{2\pi}\\ &= \int_0^{2\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{(2j+1)!}(\varphi(t+s)-\varphi(s))^{2j+1}\frac{ds}{2\pi}\,.\end{aligned}$$

Since φ is continuous we obtain

$$\Big|\sum_{-\infty}^{\infty}|a_n|^2\frac{\sin nt}{t}\Big|\le\frac{1}{|t|}\int_0^{2\pi}|\varphi(t+s)-\varphi(s)|^3\Big(\frac{1}{6}+o(1)\Big)\frac{ds}{2\pi}\ (t\to 0)\,, \tag{11}$$

therefore

$$\lim_{t\to 0}\sum_{-\infty}^{\infty}|a_n|^2\frac{\sin nt}{t}=0=\deg f \tag{12}$$

as soon as

$$\int_0^{2\pi}|\varphi(t+s)-\varphi(s)|^3ds=o(t)\qquad(t\downarrow 0)\,. \tag{13}$$

We can write (13) as

$$\int_0^{2\pi}|f(e^{i(t+s)})-f(e^{is})|^3ds=o(t)\qquad(t\downarrow 0)\,. \tag{14}$$

For (13) $\Longrightarrow$ (14) is obvious, and, since f is continuous, we have

$$|f(e^{i(t+s)})-f(e^{is})|<1$$

when t is small enough, hence

$$|\varphi(t+s)-\varphi(s)|<\frac{\pi}{3}|f(e^{i(t+s)})-f(e^{is})|\,,$$

therefore (14) $\Longrightarrow$ (13). In Zygmund's notations ([11] p. 45) (13) and (14) mean $\varphi\in\lambda^3_{1/3}$ and $f\in\lambda^3_{1/3}$ respectively. They are invariant when $f(z)$ is changed into $zf(z)$. As a conclusion:

Theorem 2.1: *If $f\in C(S^1,S^1)$ and (14) holds, that is $f\in C\cap\lambda^3_{1/3}(S^1,S^1)$, then*

$$\deg f=\lim_{t\to 0}\sum_{-\infty}^{\infty}|a_n|^2\frac{\sin nt}{t}\,. \tag{15}$$

Condition (14) is implied by

$$\int_0^{2\pi}\int_0^{2\pi}\frac{|f(e^{i(t+s)})-f(e^{is})|^3}{|e^{is}-1|^2}\,ds<\infty \tag{16}$$

the definition of $f\in W^{1/3,3}(S^1,S^1)$. On the other hand (14) is not sufficient in order to define $\deg f$, while (16) implies $f\in VMO(S^1,S^1)$ and the theory of Brézis and Nirenberg allows to define $\deg f$ (what they call $VMO-\deg f$). Theorem 5' of [3] expresses that (16) implies (15) ; let us repeat the statement.

Theorem 2.2: *(Brézis) If (16) holds, that is $f\in W^{\frac{1}{3},3}(S^1,S^1)$, then (15) holds, $\deg f$ meaning the VMO–degree of Brézis and Nirenberg.*

It is natural to try to replace $C(S^1, S^1)$ by $VMO(S^1, S^1)$ in the assumption of Theorem 2.1, I didn't succeed and I leave it as an open question for the reader.

QUESTION *Does* $f \in VMO \cap \lambda^3_{\frac{1}{3}}(S^1, S^1)$ *imply* (15) *(*$\deg f$ *meaning the* VMO*–degree) ?*

If the answer is positive, it generalizes both Theorems 2.1 and 2.2.

Here are a few comments on condition (14).

1. Let me recall the meaning of the classes Λ_α, λ_α, Λ^p_α and λ^p_α for function $g \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{C})$ $(0 < \alpha < 1,\ 1 \le p < \infty)$ ([11] pp. 42–45):

$$g \in \Lambda_\alpha : \sup_s |g(t+s) - g(s)| = O(|t|^\alpha) \qquad (t \to 0) \tag{17}$$

$$g \in \lambda_\alpha : \sup_s |g(t+s) - g(s)| = o(|t|^\alpha) \qquad (t \to 0)$$

$$g \in \Lambda^p_\alpha : \Big(\int_0^{2\pi} |g(t+s) - g(s)|^p ds\Big)^{1/p} = O(|t|^\alpha) \qquad (t \to 0)$$

$$g \in \lambda^p_\alpha : \Big(\int_0^{2\pi} |g(t+s) - g(s)|^p ds\Big)^{1/p} = o(|t|^\alpha) \qquad (t \to 0)$$

Clearly Λ^p_α resp.λ^p_α decrease and tend to Λ_α resp.λ_α as p increases and tends to ∞. All these classes decrease when α increases. Moreover, when $\alpha' > \alpha$, $\Lambda_{\alpha'}$ is included in λ_α, and $\Lambda^p_{\alpha'}$ is included in λ^p_α.

2. Here is another inclusion

$$(C \cap BV)(S^1, S^1) \subset \lambda^3_{\frac{1}{3}}(S^1, S^1)\,. \tag{18}$$

Here is a proof: assuming $f \in C(S^1, S^1)$,

$$\begin{aligned}
(14) \Longleftarrow & \int_0^{2\pi} |f(e^{i(t+s)}) - f(e^{is})|^2 ds = O(t) \quad (t \downarrow 0) \\
\Longleftrightarrow & \sum_{-\infty}^{\infty} |a_n|^2 \sin^2 \frac{t}{2} = O(t) \quad (t \downarrow 0) \\
\Longleftarrow & \; a_n = O\Big(\frac{1}{|n|}\Big) \quad (|n| \to \infty) \\
\Longleftarrow & \; f \in BV(S^1, S^1)\,.
\end{aligned}$$

3. In the opposite direction,

$$(14) \Longrightarrow \int_0^{2\pi} |f(e^{i(t+s)}) - f(e^{is})|^2 ds = o(t^{2/3}) \quad (t \downarrow 0)$$

$$\Longleftrightarrow \sum_{-\infty}^{\infty} |a_n|^2 \sin^2 \frac{t}{2} = o(t^{2/3}) \quad (t \downarrow 0)$$

$$\Longrightarrow \Big(\sum_{2^j \le |m| \le 2^{j+1}} |a_n|^2 \Big)^{1/2} = o(2^{-j/3}) \quad (j \to \infty).$$

4. Let us restrict ourselves to functions $f(e^{it}) = e^{i\varphi(t)}$ with

$$\varphi(t) = \sum_0^{\infty} \gamma_j \sin 2^j t\,. \tag{19}$$

Then we have

$$(14) \Longleftrightarrow (13) \Longleftrightarrow \gamma_j = o(2^{-j/3}) \qquad (j \to \infty) \tag{20}$$

$$\Longleftrightarrow \varphi \in \lambda_{1/3}\ (\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$$

$$\Longleftrightarrow f \in \lambda_{1/3}(S^1, S^1) \qquad \text{([8] p. 47)}$$

From now on until the end of part II, I shall restrict myself to functions φ of the form (19) and the corresponding $f(e^{it}) = e^{i\varphi(t)}$. First we observe that

$$\gamma_j = O(2^{-j/3})(j \to \infty) \Longleftrightarrow \varphi \in \Lambda_{1/3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}) \tag{21}$$

$$\Longleftrightarrow f \in \Lambda_{1/3}(S^1, S^1)\,.$$

When $\varphi \in \Lambda_{1/3}$, formula (10) gives

$$\sum_{-\infty}^{\infty} |a_n|^2 \sin nt = -\frac{1}{6} \int_0^{2\pi} (\varphi(t+s) - \varphi(s))^3 \frac{ds}{2\pi} + o(t) \quad (t \downarrow 0)\,. \tag{22}$$

We are led to compute

$$I(\{\gamma_i\}, \theta) = \int_0^{2\pi} (\varphi(s+\theta) - \varphi(s-\theta))^3 \frac{ds}{2\pi} \qquad \Big(\theta = \frac{t}{2}\Big)\,. \tag{23}$$

Starting from (19) we get

$$\varphi(s+\theta) - \varphi(s-\theta) = 2 \sum_0^{\infty} \gamma_j\ \cos 2^j\theta\ \sin 2^j s\,,$$

$$(\varphi(s+\theta) - \varphi(s-\theta))^3 = 8 \sum_{j,k,\ell} \gamma_j \gamma_k \gamma_\ell$$

$$\cos 2^j\theta\ \cos 2^k\theta\ \cos 2^\ell\theta\ \sin 2^j s\ \sin 2^k s\ \sin 2^\ell s\,.$$

Since

$$8\int_0^{2\pi}\sin 2^j s\sin 2^k s\sin 2^\ell s\ \frac{ds}{2\pi}=\begin{cases}-1 & \text{if } j=k=\ell-1 \text{ or } k=\ell=j-1 \\ & \qquad\qquad \text{or } \ell=j=k-1 \\ 0 & \text{in all other cases},\end{cases}$$

we obtain

$$I(\{\gamma_j\},\theta)=3\sum_0^\infty \gamma_j^2\gamma_{j+1}\sin^2 2^j\theta\,\sin 2^{j+1}\theta \tag{24}$$

$$=\frac{3}{2}\gamma_0^2\gamma_1\,\sin 2\theta+\frac{3}{4}\sum_{j=1}^\infty(2\gamma_j^2\gamma_{j+1}-\gamma_{j-1}^2\gamma_j)\sin 2^{j+1}\theta\,.$$

Under the condition

$$2\gamma_j\gamma_{j+1}=\gamma_{j-1}^2 \qquad (j=1,2,\ldots)$$

the last sum vanishes. Let us choose

$$\gamma_j=\gamma_0 2^{-j/3}$$

Then (21) applies, therefore $\varphi\in\Lambda_{1/3}$ and $f\in\Lambda_{1/3}$, and (22) gives

$$\sum_{-\infty}^\infty |a_n|^2\sin nt=-\frac{1}{4}\gamma_0^2\gamma_1\,\sin t+o(t) \qquad (t\to 0)\,. \tag{25}$$

Since γ_0 is arbitrary, we can state:

Theorem 2.3: *Given any real $\lambda\neq 0$, there exists $f\in\Lambda_{1/3}(S^1,S^1)$ with* $\deg f=0$ *and*

$$\lim_{t\to 0}\sum_{-\infty}^\infty |a_n|^2\frac{\sin nt}{t}=\lambda\,. \tag{26}$$

This shows that the assumption $f\in\lambda^3_{1/3}(S^1,S^1)$ in Theorem 2.1 is nearly best possible.

In formula (26) the limit exists. There are examples where it doesn't exist.

Theorem 2.4: *There exists $f\in\Lambda_{1/3}(S^1,S^1)$ such that*

$$\limsup_{t\to 0}\sum_{-\infty}^\infty |a_n|^2\frac{\sin nt}{t}=-\liminf_{t\to 0}=+\infty\,. \tag{27}$$

Proof: . We choose in (19)

$$\gamma_j=\varepsilon_j 2^{-j/3}\,,\ \varepsilon_j=\pm 1 \quad (j\in\mathbb{N})\,. \tag{28}$$

If all except a finite number of ε_j have the same sign, (24) is a trigonometric polynomial $T(\theta)$ and (26) holds with $\lambda=\frac{1}{2}T'(0)$. On the contrary, if infinitely many

ε_j are 1 and infinitely many are -1, (24) is a lacunary trigonometric series, nowhere differentiable, with upper derivative $+\infty$ and lower derivative $-\infty$, hence (27) (References can be found in [6], sections 2.8 and 6). □

3. More on summation processes

The first way to give a numerical value to the series $\sum_{-\infty}^{\infty} n|a_n|^2$ is to look for a limit of the symmetrical partial sums

$$S_n = \sum_{1}^{n} m(|a_m|^2 - |a_{-m}|^2) = \sum_{1}^{n} u_m \tag{29}$$

I already mentioned the theorem of Korevaar ([9], see also [3] p. 146): if $f \in C \cap BV(S^1, S^1)$, then

$$\deg f = \lim_{n\to\infty} S_n \,. \tag{30}$$

From now on I write as in (29)

$$u_m = m(|a_m|^2 - |a_{-m}|^2)$$

and use the fact that

$$\sum_{1}^{\infty} \frac{|u_m|}{m} < \infty \,. \tag{31}$$

Instead of the S_n I introduced

$$S(t) = \sum_{1}^{\infty} u_m \frac{\sin mt}{mt} \qquad (t \to 0)\,. \tag{32}$$

It is not a classical summation process for the series $\sum_{1}^{\infty} u_m$; it cannot be reduced to a Toeplitz method of summation ([11], chap. III). However most classical summation processes derive from it. Let us see how it works.

Consider a kernel $K(t,\varepsilon)$ $(t > 0,\ \varepsilon > 0)$ such that

$$\begin{cases} \int_0^\infty |K(t,\varepsilon)|dt < C < \infty & (\varepsilon > 0) \\ \lim_{\varepsilon\to 0} \int_0^\infty K(t,\varepsilon)dt = 1 & (a > 0) \\ \lim_{\varepsilon\to 0} \int_a^\infty K(t,\varepsilon) = 0 & (a > 0) \end{cases} \tag{33}$$

and write

$$S(K,\varepsilon) = \int_0^\infty K(t,\varepsilon)S(t)dt \,. \tag{34}$$

Proposition 3.1: .— *If* $\lim_{t\to 0} S(t)$ *exists (we always assume* (31)*), then* $\lim_{\varepsilon\to 0} S(K,\varepsilon)$ *exists and both limits are the same.*

Proof: . $\int_0^\infty = \int_0^a + \int_a^\infty$ and the second integral tends to 0. □

One can write

$$S(K,\varepsilon) = \sum_1^\infty A_m(K,\varepsilon) u_m \tag{35}$$

$$A_m(K,\varepsilon) = \int_0^\infty K(t,\varepsilon)\frac{\sin mt}{mt}dt$$

and we shall check that convenient choices of $K(\cdot,\cdot)$ give

$$A_m(K,\varepsilon) = \Big(\frac{\sin m\varepsilon}{m\varepsilon}\Big)^2 \text{ (Riemann process of summation), or}$$

$$A_m(K,\varepsilon) = e^{-m\varepsilon} \text{ (Abel-Poisson process of summation), or}$$

$$A_m(K,\varepsilon) = 1/2((1-m\varepsilon)^+ + 1\wedge\frac{1}{m\varepsilon})$$

(a substitute for the Cesàro–1 process (averages of partial sums), that uses $\Sigma(1-m\varepsilon)^+ u_m$) and, many other usual summation factors. Here is the relation with part II.

Theorem 3.2: *Theorems* 2.1, 2.2, 2.3, 2.4 *keep valid when* $\lim_{t\to 0} S(t)$ *is replaced by* $\lim_{\varepsilon\to 0} S(K,\varepsilon)$ *in their conclusion, under the assumption that the kernel* $K(\cdot,\cdot)$ *satisfies* (33).

Proof: . Obvious from the proposition for theorems 2.1, 2.2, 2.3. For theorem 2.4 one has to choose the ε_j in (28) according to $K(\cdot,\cdot)$: the idea is to take consecutive blocks of $\varepsilon_j = 1$ and consecutive blocks of $\varepsilon_j = -1$ with rapidly increasing lengths. I leave the details to the reader. □

An interesting class of kernels $K(t,\varepsilon)$ consists of the kernels $\frac{1}{\varepsilon}K\Big(\frac{t}{\varepsilon}\Big)$ and (33) becomes

$$\left\{\begin{array}{l} K \in L^1(\mathbb{R}^+) \\ \int_0^\infty K(t)dt = 1\,. \end{array}\right. \tag{36}$$

When $K(t)dt$ is the Dirac measure at point 1, one recovers

$$A_m(K,\varepsilon) = \frac{\sin m\varepsilon}{m\varepsilon}\,. \tag{37}$$

When $K(t) = 2t\ 1_{[0,1]}(t)$,

$$A_m(K,\varepsilon) = 4\,\frac{\sin^2 \frac{m}{2}\varepsilon}{m^2\varepsilon^2}\,. \tag{38}$$

In general, writing

$$\widehat{K}(u) = \int_0^\infty K(t)\cos tu\,dt\,, \tag{39}$$

and

$$A_m(K,\varepsilon) = \frac{1}{\varepsilon m}\int_0^\infty K(t)\sin\varepsilon mt\frac{dt}{t} \tag{40}$$

we can derive $K(\cdot)$ from the formula

$$\frac{d}{d\varepsilon}(\varepsilon A_m(K,\varepsilon)) = \widehat{K}(\varepsilon m)\,. \tag{41}$$

This applies to the examples $e^{-m\varepsilon}$ and $1/2((1-m\varepsilon)^+ + 1\wedge\frac{1}{m\varepsilon})$ that I gave before. It does not apply to the more natural example $A_m(K,\varepsilon) = (1-m\varepsilon)^+$ (Cesàro–1 process) because the left hand–side of (41) is discontinuous in that case.

It is not possible to derive the convergence of the S_n ($n\to\infty$) from the convergence of $S(t)$ ($t\to 0$). There exist functions $\sum_1^\infty \frac{u_m}{m}\sin mt$ differentiable at 0 with $\sum_1^\infty u_m$ divergent: for there are classical examples of continuous functions $\sum_1^\infty u_m\cos mt$ with divergent Fourier series at 0 ([11] chap. VIII).

This derivation needs an extra condition on the u_m. It works when $u_m = O\left(\frac{1}{m}\right)$ ($m\to\infty$). Actually the Tauberian theorem of Littlewood ([11], p. 81) asserts that, under this condition, the Abel–Poisson summability of the series $\sum_1^\infty u_m$ implies that it converges, and we just saw that the convergence of $S(t)$ ($t\to 0$) implies the Abel–Poisson summability.

Since $f\in BV$ implies $a_n = O\left(\frac{1}{n}\right)$, which in turn implies $u_m = O\left(\frac{1}{m}\right)$, that is a (rather complicated) way to recover Korevaar's theorem, that $f\in C\cap BV(S^1,S^1)$ implies $\deg f = \lim_{n\to\infty} S_n$.

Let me go on with a puzzle: to look for a sequence $(u_m)_{m\in\mathbb{N}}$ such that $\lim_{n\to\infty} S_n$ exists and $\lim_{t\to 0} S(t)$ doesn't.

Here is a solution. Choose

$$t_j = \exp(-\exp j^2)\,,\quad n_j = \left[\frac{1}{t_j}\right],$$

and

$$S_n = u_1 + u_2 + \cdots + u_n = \frac{1}{\log n}\,sign\cos nt_j$$

when $n_j \le n < n_{j+1}$. Then one can check that

$$\sum_{n_j \le m < n_{j+1}} \frac{|u_m|}{m} \le 10 \ t_j \sum_{n_j \le n < n_{j+1}} \frac{|S_n|}{n} \le 100 j t_j \,.$$

therefore

$$\sum_1^\infty \frac{|u_m|}{m} < \infty \,,$$

and

$$\begin{aligned} t_j &= \sum_{n < n_{j+1}} S_n \int_n^{n+1} \Big(\frac{\sin xt}{xt} - \cos xt\Big) \frac{dx}{x} + O(1) \\ &= -\int_{n_j}^{n_{j+1}} S_{n(x)} \cos xt \frac{dx}{x} + O(1) \qquad (j \to \infty) \end{aligned}$$

where $S_{n(x)} = S_n$ when $n \le x < n+1$, therefore

$$\lim_{j\to\infty} t_j = -\infty$$

and $\lim\limits_{t\to 0} S(t)$ doesn't exist, while $\lim\limits_{n\to\infty} S_n = 0$.

This solution took advantage of an oral and e-mail communication with Zoltan Buczolich, who has constructed another example.

Added in proofs

The first version of this article contained a mistake. I thought that the Cesàro–1 means of the u_m converge to $\lim\limits_{t\to 0} S(t)$ (defined in (32)) when this limit exists, and I gave $A_m(K,\varepsilon) = (1 - m\varepsilon)^+$ as an application of (41). Quite politely, the referee asked me to explain the point. Actually there is no explanation, but a rectification. I had to change $(1-m\varepsilon)^+$ into $1/2((1-m\varepsilon)^+ + 1 \wedge \frac{1}{m\varepsilon})$. Here is the reason.

Proposition 3.3: *There exists a sequence* (u_m) $(m \in \mathbb{N})$ *which satisfies* (31) *and* $\lim\limits_{t\to 0} S(t) = 0$ *(*$S(t)$ *defined in* (32)*), such that the Cesàro–1 means* $\sum\limits_1^N (1 - \frac{m}{N}) u_m$ *diverge* $(N \to \infty)$.

Proof: From the rectification we know that

$$\frac{1}{2}\Big(\sum_0^N (1 - \frac{m}{n}) u_m + \sum_0^\infty (1 \wedge \frac{N}{m}) u_m\Big)$$

has the same limit $(N \to \infty)$ as $S(t)$ $(t \to 0)$. I shall choose the u_m in such a way that $\lim\limits_{t\to 0} S(t) = 0$ and $\sum\limits_0^\infty (1 \wedge \frac{N}{m}) u_m$ diverge. The last condition is satisfied if the integrals

$$\int_{-\pi}^{\pi} \sin Nt \ S(t) \frac{dt}{t}$$

are unbounded. We define

$$S(t) = \sum_{j=1}^{\infty} \varphi_j(t) \sin\ N_j t\,,$$

the φ_j being "triangle" functions with disjoint supports, vanishing at 0, such that $\sum_1^\infty \|\varphi_j\|_{A(\mathbb{T})} < \infty$, $A(\mathbb{T}) = \mathcal{F}\ell^1(\mathbb{Z})$ denoting the Wiener algebra, and $\lim_{j\to\infty} \int \varphi_j(t)\frac{dt}{t} = \infty$, and the N_j being chosen rapidly increasing, in such a way that

$$\lim_{j\to\infty} \int \sin\ N_j t\ S(t)\frac{dt}{t} = \infty\,.$$

Since $\|t\ S(t)\|_{A(\mathbb{T})} < \infty$, (31) is satisfied. □

Here is the context. The book of Hardy "Divergent series" [7] contains on Appendix where three types of summability processes are compared : (R,k), (C,k) and (A). (A) means Abel, (C,k) is the Cesàro process of order k, and (R,k) is defined by the multipliers $(\frac{\sin\ mt}{mt})^k$: $(R,2)$ is the original Riemann process and $(R,1)$ is the non–standard process considered in this paper.

Hardy established and quoted a number of results and refers to Kuttner 1935 [10] for

$$(R,2) \Longrightarrow (C,2+\delta) \text{ and } (R,1) \Longrightarrow (C,1+\delta) \quad (\delta > 0)\,.$$

Actually the second implication is already in Zygmund 1928 [12]. Zygmund mentioned that Privaloff gave an example of a series $(C,1,\delta)$ summable for all $\delta > 0$ but not for $\delta = 0$. The proposition above means

$$(R,1) \not\Rightarrow (C,1)$$

and it is also a way to recover Privaloff's result.

Kuttner considered also $(R,3)$, and showed that

$$(R,3) \not\Rightarrow (A)\,.$$

References

[1] J. Bourgain et J.–P. Kahane. *Sur les séries de Fourier des fonctions continues unimodulaires (2009)*, submitted to Annales de l'Institut Fourier.

[2] J. Bourgain and G. Kozma. *One cannot hear the winding number*, J. Europ. Math. soc. 9 (2007), 637–658.

[3] H. Brézis. *New questions related to the topological degree*, The unity of mathematics, in honour of I.M. Gelfand, Progress in Mathematics 244 (2006), 137–154.

[4] H. Brézis. *Oral communication*, Conference NODE in honour of J. Mawhin and J. Habetz, Bruxelles, September 2008.

[5] H. Brézis and J. Nirenberg. *Degree theory and BMO*, Part I, Selecta mathematica 1 (1995), 197–263.

[6] J. Dixmier, J.–P. Kahane et J.–L. Nicolas. *Un exemple de non–dérivabilité en géométrie du triangle*, L'Enseignement mathématique 53 (2007), 369–428.

[7] G.H. Hardy. *Divergent series*, Oxford 1949, New–York 1991.

[8] J.–P. Kahane. *Sur l'équation fonctionnelle* $\int_\pi (\psi(t+s)-\psi(s))^3 ds = sint$, C.R. Math. Acad. Sc. Paris 341 (2005), 141–145.

[9] J. Korevaar. *On a question of Brezis and Nirenberg concerning the degree of circle maps*, Selecta mathematica 5 (1999), 107–122.

[10] B. KUTTNER. *The relation between Riemann and Cesàro summability*, Proc. London Math. Soc. (2) 38 (1935), 273–283.
[11] A. ZYGMUND. *Trigonometric series I*, Cambridge University Press 1959.
[12] A. ZYGMUND. *Sur la dérivation des séries de Fourier*, Bull. Acad. Polon. 1924, 243–249.