# The Souriau-Hsu-Möbius (SHM) function

October 2009

# Introduction

Let $\mathcal{A}$ be the set of arithmetic functions equipped with addition, multiplication and Dirichlet convolution defined over $\mathbb{N}$, respectively, by

$$(f+g)(n) = f(n) + g(n), \ (fg)(n) = f(n)g(n),$$

$$(f * g)(n) = \sum_{ij=n} f(i)g(j) \quad (n \in \mathbb{N}).$$

The convolution identity $I \in \mathcal{A}$ is defined by

$$I(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

For $f \in \mathcal{A}$, write $f^{-1}$ for its convolution inverse whenever it exists. A function $f \in \mathcal{A} \setminus \{0\}$ is said to be *multiplicative* if

$$f(mn) = f(m)f(n) \text{ whenever } \gcd(m, n) = 1,$$

and is called *completely multiplicative* if this equality holds for all $m, n \in \mathbb{N}$.
Denote by $\mathcal{M}, \mathcal{C}$ the sets of all multiplicative and completely multiplicative functions, respectively.

The classical Möbius function is defined by $\mu(1) = 1$ and for $n = p_1^{a_1} \cdots p_k^{a_k}$ by

$$\mu(n) = (-1)^k \text{ if } a_1 = \cdots = a_k = 1, \text{ and 0 otherwise.}$$

The Souriau-Hsu-Möbius (SHM) function, $\mu_\alpha$ is defined by

$$\mu_\alpha(n) = \prod_{p|n} \binom{\alpha}{\upsilon_p(n)} (-1)^{\upsilon_p(n)}$$

where $\alpha \in \mathbb{C}$ , and $n = \prod p^{\upsilon_p(n)}$ denotes the unique prime factorization of $n \in \mathbb{N}$, $\upsilon_p(n)$ being the largest exponent of the prime $p$ that divides $n$.

This function was introduced by Souriau in 1944 (Jean-Marie Souriau, Généralisation de certaines formules arithmétiques d'inversion. Applications, Revue Scientific (Rev. Rose Illus.) **82**(1944), 204-211). It generalizes the usual Möbius function, $\mu$, because $\mu_1 = \mu$. Note that

$$\mu_0 = I, \text{ the convolution identity,}$$
$$\mu_{-1} = u, \text{ the arithmetic unit function defined by } u(n) = 1 \ (n \in \mathbb{N}).$$

There are exactly two SHM functions that are completely multiplicative, namely, $\mu_0 = I$ and $\mu_{-1} = u$, and there is exactly one SHM function whose convolution inverse is completely multiplicative, namely, $\mu_1 = u^{-1}$.

The well-known Möbius inversion formula states that

$$g(n) = \sum_{a|n} f(a) \iff f(n) = \sum_{a|n} \mu\left(\frac{n}{a}\right) g(a),$$

which is equivalent to saying that $g = u * f \iff f = \mu * g$.

D. Rearick, *Operators on algebras of arithmetic functions,* Duke Math. J. **35** (1968),761-766; D. Rearick, *The trigonometry of numbers,* Duke Math. J. **35** (1968),767-776.
The log-derivation is the operator $D : \mathcal{A} \to \mathcal{A}$ defined by

$$(Df)(n) = f(n) \log n \quad (n \in \mathbb{N}).$$

For $f \in \mathcal{A}$, $f(1) > 0$, the Rearick logarithmic operator of $f$ (or logarithm of $f$), denoted by $\mathrm{Log}\, f \in \mathcal{A}$, is defined via

$$(\mathrm{Log}\, f)(1) = \log f(1),$$

$$(\mathrm{Log} f)(n) = \frac{1}{\log n} \sum_{d|n} f(d) f^{-1}\left(\frac{n}{d}\right) \log d = \frac{1}{\log n} (Df * f^{-1})(n) \quad (n > 1),$$

where $D$ denotes the log-derivation.

For $h \in \mathcal{A}$, the Rearick exponential Exp $h$ is defined as the unique element $f \in \mathcal{A},\ f(1) > 0$ such that $h = \mathrm{Log}\, f$. For $f \in \mathcal{A},\ f(1) > 0$ and $\alpha \in \mathbb{R}$, the $\alpha^{th}$ power function is defined as

$$f^{\alpha} = \mathrm{Exp}\,(\alpha\, \mathsf{Log}\, f);$$

this agrees with the usual power function when $\alpha$ is integral. We know that if $f$ is multiplicative and $\alpha \in \mathbb{R} \setminus \{0\}$, so is $f^{\alpha}$.

# Souriau's 1944 results

1. $\mu_\alpha$ $(\alpha \in \mathbb{C})$ is multiplicative;
2. $\mu_{\alpha+\beta} = \mu_\alpha * \mu_\beta \quad (\alpha, \beta \in \mathbb{C})$.
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\mu_\alpha(n)}{n^s} = \zeta(s)^\alpha \quad (\alpha \in \mathbb{C})$.

# Brown-Hsu-Wang-Shiue's 2000 results

T.C. Brown, L.C. Hsu, J. Wang and P.J.-S. Shiue, On a certain kind of generalized number-theoretical Möbius function, Math. Sci. **25**(2000), no.2, 72-77.
J. Wang and L.C. Hsu, On certain generalized Euler-type totients and Möbius-type functios, Dalian University of Technology, China, preprint.

1. The set $M := \{\mu_\alpha \ : \ \alpha \in \mathbb{C}\}$ is an abelian group with respect to the convolution $*$;
2. For $\alpha \in \mathbb{C}$, we have the inversion formula $f = \mu_\alpha * g \Longleftrightarrow g = \mu_{-\alpha} * f$;

# LPW 2002

1. The arithmetic function $f$ is comletely multiplicative if and only if $(\mu_\alpha f)^{-1} = \mu_{-\alpha} f$.
2. Assume the condition: if $\alpha$ is a negative even integer, then $f(p^{-\alpha-1}) = f(p)^{-\alpha-1}$ for each prime $p$. If $f^\alpha = \mu_{-\alpha} f$, then $f$ is completely multiplicative.

Classical: Let $f \in \mathcal{M}$. Then $f \in \mathcal{C} \iff f^{-1} = \mu f$.

# LP 2004

1. Let $f \in \mathcal{A}$ be multiplicative, $\alpha \in \mathbb{R} \setminus \{0\}$. Then $f$ is completely multiplicative if and only if

$$f^{\alpha}(p^{a}) = \mu_{-\alpha}(p^{a}) \left( \frac{1}{\alpha} f^{\alpha}(p) \right)^{a}$$

for all primes $p$ and all $a \in \mathbb{N}$.

Classical: Let $f \in \mathcal{M}$. Then $f \in \mathcal{C} \Longleftrightarrow f^{-1}(p^{a}) = 0$ for each prime $p$ and $a \geq 2$.

# PLR 2005

Let $\zeta_k(n) = n^k$. Let $s \in \mathbb{N} \cup \{0\}$. Define

$$\mathcal{C}_s := \{f \in \mathcal{M}; \text{there exists } a(p) \in \mathbb{C}\backslash\{0\}$$
$$\text{such that } f(p^{s+j}) = a(p)^j f(p^s) \ (p \ \text{ prime}, j \in \mathbb{N} \cup \{0\})\}.$$

Recall that a positive integer is said to be d-powerful, $d \in \mathbb{N}$, if it is divisible by each of its prime factors up to a power of $d$.

# SHM-Landau identity

Let $\alpha \in \mathbb{C}$, $f \in \mathcal{A}$ be multiplicative and $\beta \in \mathbb{N}$. Assume for each prime $p$ and $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ that there exists $a(p) \in \mathbb{C}\backslash\{0\}$ such that $f(p^{\beta+j}) = i_{\beta,a}(p^j)f(p^\beta)$, where $i_{\beta,a}(p^j) := a(p)^j$. Then for $r$ being $\beta$-powerful, we have

$$\sum_{d|r} \frac{(\mu_\alpha(d))^2}{f(d)} = \prod_{p|r} \left[ \sum_{m=0}^{\beta-1} \frac{\binom{\alpha}{m}^2}{f(p^m)} + \frac{1}{f(p^\beta)} \sum_{i=\beta}^{\nu_p(r)} \frac{\binom{\alpha}{i}^2}{a(p)^{i-\beta}} \right]$$

$$= \prod_{p|r} \left[ \sum_{m=0}^{\beta-1} \frac{\binom{\alpha}{m}^2}{f(p^m)} + \frac{(i_{\beta,a} * (\mu_\alpha)^2)(p^a)}{f(p^a)} - \frac{a(p)(i_{\beta,a} * (\mu_\alpha)^2)(p^{\beta-1})}{f(p^\beta)} \right].$$

# SHM-Brauer-Radamacher identity

Let $n, r \in \mathbb{N}, s \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{C}$. If $f \in \mathcal{C}_s$ and $g \in \mathcal{C}$, then

$$f(r) \sum_{\substack{d|r \\ (d,n)=1}} \frac{g(d)}{f(d)} \mu_\alpha\left(\frac{r}{d}\right) = \prod_{\substack{p|r \\ p\nmid n}} \left\{ f(p^{\nu_p(r)}) \sum_{m=0}^{s-1} \frac{g(p)^m}{f(p^m)} \binom{\alpha}{\nu_p(r)-m} (-1)^{\nu_p(r)-} \right.$$

$$\left. \sum_{l=0}^{\nu_p(r)-s} a(p)^{\nu_p(r)-s-l} g(p)^{s+l} \binom{\alpha}{\nu_p(r)-s-l} (-1)^{\nu_p(r)-s-l} \right\} \prod_{\substack{p|r \\ p|n}} (f\mu_\alpha)(p^{\nu_p(r)}),$$

where $r$ is $s$-powerful and $a(p)$ is as defined in $\mathcal{C}_s$.

# Independence

- Let $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \ldots, \beta_k \in \mathbb{C}$ such that $\beta_i \neq 0$ and $\beta_i \neq \beta_j \;\; (i \neq j)$. Then $\alpha_1, \ldots, \alpha_r$, are linearly independent over $\mathbb{Z}$ if and only if $\mu_{\alpha_1}, \ldots, \mu_{\alpha_r}, \zeta_{\beta_1}, \ldots, \zeta_{\beta_k}$ are algebraically independent over $\mathbb{C}$.
- Let $\alpha, \beta \in \mathbb{C}$. If $\alpha$ and $\beta$ are linearly independent over $\mathbb{Z}$, then $\mu_\alpha$ and $\mu_\beta$ are algebraically independent over $\mathbb{C}$.

• Let $\alpha, \beta \in \mathbb{C}$ be linearly independent over $\mathbb{Z}$.
(i) If a nonzero multiplicative function $f$ is algebraic over $\mathbb{C}[\mu_\alpha]$, then there exists $r \in \mathbb{Q}$ such that

$$f = \mu_\alpha^r = \mu_{r\alpha}.$$

(ii) If a nonzero multiplicative function $f$ is algebraic over $\mathbb{C}[\mu_\alpha, \mu_\beta]$, then there are $r, t \in \mathbb{Q}$ such that

$$f = \mu_\alpha^r * \mu_\beta^t = \mu_{r\alpha+t\beta}.$$

# LRP 2006

The Generalized Ramanujan sum of order $\alpha \in \mathbb{C}$ is defined by

$$c^{(\alpha)}(n,k) := \sum_{d \mid \gcd(n,k)} d\mu_\alpha\left(\frac{k}{d}\right) \quad (n \in \mathbb{N}_0,\ k \in \mathbb{N});$$

when $\alpha = 1$, we get back to the usual Ramanujan sum.

$$c^{(\alpha)}(1, m) = \mu_\alpha(m)$$

$$c^{(\alpha)}(n, m) = \mu_\alpha(m) \text{ if } \gcd(n, m) = 1.$$

$$c^{(\alpha)}(a, mk) = c^{(\alpha)}(a, m)\mu_\alpha(k) \text{ if } \gcd(a, k) = \gcd(m, k) = 1$$

# Dependence

- Let $k \in \mathbb{N}$, $n \in \mathbb{N}_0$ be fixed integers, and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$. If $\alpha_1, \ldots, \alpha_k$ are $\mathbb{Z}$-linearly independent, then $c^{(\alpha_1)}(n, \cdot), \ldots, c^{(\alpha_k)}(n, \cdot)$ are $\mathbb{C}$-algebraically independent.
- Let $k \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$. If $c^{(\alpha_1)}(1, \cdot), \ldots, c^{(\alpha_k)}(1, \cdot)$ are $\mathbb{C}$-algebraically independent, then $\alpha_1, \ldots, \alpha_k$ are $\mathbb{Z}$-linearly independent.

An Euler-type totient is defined as an arithmetic function of the form

$$\phi_k^{(\alpha)} := \zeta_k * \mu_\alpha \ (\alpha \in \mathbb{C}, \ k \in \mathbb{Z}).$$

Thus, for $r|n$, we also have $c^{(\alpha)}(n,r) = \phi_1^{(\alpha)}(r)$.

The mean value of an arithmetic function $f$ is defined by

$$M(f) := \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n),$$

whenever this limit exists.

- If $k \in \mathbb{N}, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, then

$$\sum_{d \mid \gcd(n,k)} d\mu_\alpha\left(\frac{k}{d}\right) = \sum_{m \pmod{k}} \mu_{\alpha-1}(\gcd(m,k))e^{2\pi mni/k}.$$

- If $h(p) \neq 0$ for all primes $p$ and $H = 1/h$ , then for any $n, r \in \mathbb{N}$,

$$F_\alpha(r) \sum_{\substack{d \mid r \\ \gcd(n,d)=1}} \frac{h(d)}{F_\alpha(d)}\mu_\alpha\left(\frac{r}{d}\right)$$

$$= \prod_{p \mid n}(F_\alpha\mu_\alpha)(p^{\upsilon_p(r)}) \prod_{p \nmid n}\left(F_\alpha h\left(\frac{1}{F_\alpha} * H^{-\alpha}\right)\right)(p^{\upsilon_p(r)}).$$

• Assume that $h(p) \neq 0$ for all primes $p$. Let $\langle n, r \rangle$ be a unitary pair and $N = \frac{r}{\gcd(n,r)}$. Then

$$f_\alpha(n, r) = \frac{g(N)\mu_\alpha(N)F_\alpha(r)}{F_\alpha(N)},$$

whenever $F_\alpha(N) \neq 0$, where $F_\alpha(r) = (h * g\mu_\alpha)(r)$.

• (Orthogonality?) Let $\alpha, \beta \in \mathbb{C}$ and $n, r \in \mathbb{N}$. If $r$ is divisible by both $s$ and $t$, then

$$\sum_{a+b\equiv n \pmod r} c^{(\alpha)}(a,s)c^{(\beta)}(b,t)$$
$$= r\sum_{k=0}^{q-1} \mu_{\alpha-1}(s_1 \gcd(k,g))\mu_{\beta-1}(t_1 \gcd(k,g))e^{2\pi ikn/g},$$

where $g = \gcd(s,t),\ s = gs_1,\ t = gt_1,\ \gcd(s_1,t_1) = 1.$

- If $\alpha \in \mathbb{C}$ and $m, s \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} \frac{\mu_\alpha(mn)}{n^s} = \frac{\mu_\alpha(m)}{\zeta^\alpha(s)} \prod_{p|m} \frac{1 + \sum_{i\geq 1} \frac{\mu_\alpha(p^{\upsilon_p(m)+i})}{\mu_\alpha(p^{\upsilon_p(m)})p^{is}}}{1 + \sum_{i\geq 1} \frac{\mu_\alpha(p^i)}{p^{is}}}.$$

- For a fixed $r \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, we have $M(c^{(\alpha)}(\cdot, r)) = \mu_{\alpha-1}(r)$.

- If $\alpha, n \in \mathbb{N}$ and $s > 0$ is real, then

$$\frac{\phi_s^{(\alpha)}(n)\left(\zeta(s+1)\right)^{\alpha}}{n^s} = \sum_{j=1}^{\infty} \frac{c(n,j)\mu_\alpha(j)}{j^{s+1}} \prod_{p|j} \frac{1 + \sum_{i\geq 1} \frac{\mu_\alpha(p^{\upsilon_p(j)+i})}{\mu_\alpha(p^{\upsilon_p(j)})p^{i(s+1)}}}{1 + \sum_{i\geq 1} \frac{\mu_\alpha(p^i)}{p^{i(s+1)}}}.$$

- If $n, r$ and $\alpha \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} \frac{c^{(\alpha)}(n,r)}{n} = -\sum_{d|n} \mu_\alpha\left(\frac{r}{d}\right) \log d,$$

whenever $r$ is $\alpha$-powerful.

- For a fixed $\alpha \in \mathbb{C}$, and $m, n, k \in \mathbb{N}$, if

$$M^{(\alpha)}(m,n) := \sum_{r \mid \gcd(m,n)} r\mu_\alpha\left(\frac{m}{r}\right)\mu_{\alpha-1}\left(\frac{n}{r}\right)$$

and

$$g_k^{(\alpha)}(n) := \frac{1}{n}\sum_{r\geq 1}\frac{\mu_\alpha(r)}{r}(\log(nr))^k,$$

then

$$\sum_{n\geq 1}\frac{M^{(\alpha)}(m,n)}{n}(\log(n))^k = \sum_{n\geq 1} c^{(\alpha)}(n,m)g_k^{(\alpha)}(n).$$

# Even functions

For a fixed $r \in \mathbb{N}$, an arithmetic function $f$ is called an **even function (mod $r$)** if $f(\gcd(n,r)) = f(n)$ for all $n \in \mathbb{N}$. It is well-known that an even function (mod $r$), $f$, is uniquely representable in the form

$$f(n) = \sum_{d|r} a(d)c(n,d) \qquad (n \in \mathbb{N}),$$

where the **Fourier coefficients** $a(d)$ are given by

$$a(d) = \frac{1}{r}\sum_{e|r} f\left(\frac{r}{e}\right) c\left(\frac{r}{d}, e\right) = \frac{1}{r\phi(d)} \sum_{m=1}^{r} f(m)c(m,d).$$

An even function (mod $r$) $f$ can be uniquely expanded as a Fourier expansion in the form

$$f(n) = \sum_{D|r} A^{(\alpha)}(D) c^{(\alpha)}(n, D),$$

where the Fourier coefficients $A^{(\alpha)}(D)$ are given by

$$A^{(\alpha)}(D) = \sum_{j|\frac{r}{D}} a(jD) A(jD; D),$$

with

$$a(d) = \frac{1}{r} \sum_{e|r} f\left(\frac{r}{e}\right) c\left(\frac{r}{d}, e\right) = \frac{1}{r\phi(d)} \sum_{m=1}^{r} f(m)c(m,d),$$

and the A(jD;D) are uniquely determined from

$$\sum_{k|\frac{t}{d}} A(t;kd) a_{kd}^{(\alpha)}(d) = \begin{cases} 0 & \text{if } \ d \neq t, d|t \\ 1 & \text{if } \ d = t, \end{cases}$$

with $a_{kd}^{(\alpha)}(d)$ defined by

$$a_r^{(\alpha)}(d) = \frac{1}{r} \sum_{e|r} c^{(\alpha)}\left(\frac{r}{e}, r\right) c\left(\frac{r}{d}, e\right) = \frac{1}{r\phi(d)} \sum_{m=1}^{r} c^{(\alpha)}(m,r)c(m,d).$$

# Completely multiplicativity

Let $f$ be a non-zero multiplicative function whose associated Dirichlet series is $F(s) = \sum_{n\geq 1} \frac{f(n)}{n^s}$. Let $c^{(\alpha)}(m,n)$ be the GRS of order $\alpha \in \mathbb{C}$.

(i) If $f$ is completely multiplicative , then for each $m \in \mathbb{N}$

$$F(s) \sum_{n\geq 1} \frac{f(n)c^{(\alpha)}(m,n)}{n^s} = \sum_{n\geq 1} \frac{c^{(\alpha-1)}(m,n)}{n^s} f(n). \qquad (1)$$

(ii) If equation (1) holds for all $m \in \mathbb{N}$, then $f$ is completely multiplicative.

# Some combinatorics

The classical *Euler's totient* $\phi(n)$ is defined as the number of positive integers $a \leq n$ such that $\gcd(a, n) = 1$. It is well known that

$$\phi(n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right) = n\prod_{p|n}\left(1 - \frac{1}{p}\right).$$

A generalized Euler totient (GET) is defined as

$$\phi^{f}_{s,\alpha}(n) := (\zeta_s * \mu_\alpha f)(n) = \sum_{d|n}\left(\frac{n}{d}\right)^s f(d)\mu_\alpha(d),$$

where $\alpha \in \mathbb{C}$, $s \in \mathbb{R}$ and $\zeta_s(n) = n^s$, $\zeta_0 = u$. For brevity write

$$\phi_{s,\alpha} := \phi^{u}_{s,\alpha} \quad \text{and} \quad \phi_\alpha := \phi_{1,\alpha}.$$

When the parameters $s$ and $\alpha$ take integer values, the GET does indeed represent a number of well-known arithmetic functions, namely,

$$\phi_{1,1} = \zeta * \mu = \phi \text{ (the classical Euler totient)}$$
$$\phi_{0,-1} = u * u = \sigma_0 = \tau \text{ (the number of divisors function)}$$
$$\phi_{s,-1} = \zeta_s * u = \sigma_s \text{ (the sum of the } s^{th} \text{ power of divisors function.}$$

When $s \in \mathbb{N}$ and $\alpha = 1$, this particular totient

$$\phi_{s,1} = \zeta_s * \mu$$

is equivalent to quite a few classical totients, namely,
(i.1) the Jordan totient $J_s(n)$ which counts the number of $s$-tuples $(x_1, \ldots, x_s) \in \mathbb{Z}_n^s$ such that $\gcd(x_1, \ldots, x_s, n) = 1$;

(i.2) the von Sterneck function

$$H_s(n) := \sum_{\mathrm{lcm}(e_1,\ldots,e_s)=n} \phi(e_1)\cdots\phi(e_s),$$

where the sum is over all ordered $s$-tuples $(e_1,\ldots,e_s) \in \mathbb{Z}^s$ such that $1 \leq e_i \leq n \quad (i = 1,\ldots,s)$ and lcm $(e_1,\ldots,e_s) = n$.

(i.3) the Eckford Cohen's totient $E_s(n)$ which counts the number of elements of a $s$-reduced residue system (mod $n$). For integers $a, b$ not both 0, let $(a, b)_s$ denote the largest $s^{th}$-power common divisor of $a$ and $b$. If $(a, b)_s = 1$, we say that $a$ and $b$ are relatively $s$-prime. We refer to the subset of a complete residue system $M$ (mod $n^s$) consisting of all elements of $M$ that are relatively $s$-prime to $n^s$ as a $s$-reduced residue system (mod $n$);

(i.4) $\phi_{s,1}(n) = \Phi_s(n^s)$, where $\Phi(n)$ is the Klee's totient which counts the number of integers $h \in \{1, 2, \ldots, n\}$ for which $\gcd(h, n)$ is $s^{th}$-power-free, i.e, contains no $s^{th}$-power divisors other than 1.

# Thank You