Title :

On the elementary solution of the operator $\otimes_B^k$

Author :

Eakachai Suntonsinsoungvon
and
Amnuay Kananthai

Institution :

Department of Mathematics, Faculty of Science,
Chiang Mai University, Chiang Mai, 50200,
Thailand

Abstract.

In this article, we study the elementary solution of the form $\otimes_B^k$ which is iterated k-times and is defined by

$$\otimes_B^k = \left[\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^3 - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)^3\right]^k,$$

where $p+q=n$ is the dimension of $\mathbb{R}_n^+$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \ldots, n$ and k is a positive integer. At first, we study the elementary solution of the operator $\otimes_B^k$ and after that, we apply such an elementary solution to solve for the solution of the equation $\otimes_B^k u(x) = f(x)$, where f is a generalized function and u is an unknown function.

Introduction.

Kananthai [3] has first introduced the Diamond operator and has proved that the convolution solution $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is an unique elementary solution of the equation $\diamondsuit^k u(x) = \delta$, where $\diamondsuit^k$ is the Diamond operator iterated k times and is defined by

$$\diamondsuit^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k , \qquad (1)$$

where $p + q = n$ is the dimension of the space $\mathbb{R}^n$ and k is a nonnegative integer. Actually the Diamond operator can be expressed as the product of the Laplace operator and the ultra-hyperbolic operator, that is

$$\diamondsuit^k = \triangle^k \Box^k = \Box^k \triangle^k$$

where $\triangle^k$ is the Laplace operator iterated k times, defined by

$$\triangle^k = \left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \right)^k , \tag{2}$$

and $\Box^k$ is the ultra-hyperbolic operator iterated k times with $p + q = n$, defined by

$$\Box^k = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k . \tag{3}$$

Furthermore, Yildirim et al. [9] have introduced the Bessel diamond operator and have proved that the convolution solution $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is an unique elementary solution of the equation $\diamondsuit_B^k u(x) = \delta$, where $\diamondsuit_B^k$ is the Bessel operator iterated k times with $x \in \mathbb{R}_n^+$,

$$\diamondsuit_B^k = \left[\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^2 - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)^2\right]^k, \quad (4)$$

$p + q = n$ is the dimension of $\mathbb{R}_n^+$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \ldots, n$ and k is a positive integer.

The Bessel diamond operator $\Diamond_B$ can be expressed in the form $\Diamond_B = \Box_B \triangle_B = \triangle_B \Box_B$, where $\triangle_B$ is the Laplace-Bessel operator, defined by

$$\triangle_B = B_{x_1} + B_{x_2} + \cdots + B_{x_n}, \tag{5}$$

and $\Box_B$ is the Bessel ultra-hyperbolic operator, defined by

$$\Box_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - B_{x_{p+2}} - \cdots - B_{x_{p+q}}. \tag{6}$$

In this research, at first we study the elementary solution of the operator $\otimes_B^k$, that is

$$\otimes_B^k G(x) = \delta, \tag{7}$$

where $G(x)$ is the elementary solution of such equation, δ is the Dirac delta distribution, k is nonnegative integer and the operator $\otimes_B$ is defined by

$$\begin{aligned} \otimes_B &= \left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^3 - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)^3 \\ &= \frac{3}{4}\Diamond_B \triangle_B + \frac{1}{4}\Box_B^3. \end{aligned} \tag{8}$$

After that, we apply such an elementary solution to solve for the solution of the equation $\otimes_B^k G(x) = f(x)$, where $f(x)$ is a given generalized function and $u(x)$ is an unknown function.

Preliminaries

Definition 1. The Fourier-Bessel transformation and its inverse transformation are defined as follows [8],

$$(F_B f)(x) = C_v \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} \right) dy,$$

$$(F_B^{-1} f)(x) = (F_B f)(-x), \; C_v = \left(\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right) \right)^{-1},$$

where $J_{v_i-\frac{1}{2}}(x_i y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. There are following equalities for Fourier-Bessel transformation [8],

$$F_B \delta(x) = 1 \text{ and } F_B(f * g)(x) = F_B f(x) \cdot F_B g(x).$$

Lemma 2. *There is a following equality for Fourier-Bessel transformation*

$$F_B(|x|^{-\alpha}) = 2^{n+2|v|-2\alpha}\Gamma\left(\frac{n+2|v|-\alpha}{2}\right)\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^{-1}|x|^{\alpha-n-2|v|}$$

where $|v| = v_1 + v_2 + \cdots + v_n$.

The proof of this Lemma is given in [8].

Lemma 3. *Given the equation* $\triangle_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$*, where* $\triangle_B^k$ *is the Laplace-Bessel operator iterated* k*-times defined by* (5)*. Then* $u(x) = (-1)^k S_{2k}(x)$ *is an elementary solution of the operator* $\triangle_B^k$*, where*

$$S_{2k}(x) = \frac{2^{n+2|v|-4k}\Gamma(\frac{n+2|v|-2k}{2})}{\Pi_{i=1}^{n} 2^{v_i - \frac{1}{2}}\Gamma(v_i + \frac{1}{2})\Gamma(k)} |x|^{2k-n-2|v|}. \qquad (9)$$

The proof of this Lemma is given in [9].

Lemma 4. *Given the equation* $\square_B^k u(x) = \delta(x)$ *for* $x \in \Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \cdots, x_n > 0$ *and* $V > 0\}$, *where* $\square_B^k$ *is the Bessel-ultra-hyperbolic operator iterated* k*-times defined by* (6). *Then* $u(x) = R_{2k}(x)$ *is an elementary solution of the operator* $\square_B^k$, *where*

$$R_{2k}(x) = \frac{V^{\frac{2k-n-2|v|}{2}}}{K_n(2k)} \tag{10}$$

for $V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ *and*

$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}}\Gamma(\frac{2+2k-n-2|v|}{2})\Gamma(\frac{1-2k}{2})\Gamma(2k)}{\Gamma(\frac{2+2k-p-2|v|}{2})\Gamma(\frac{p-2k}{2})}$$

The proof of this Lemma is given in [9].

Lemma 5. *Given the equation* $\Diamond_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$*, where* $\Diamond_B^k$ *is the diamond Bessel operator iterated* k*-times defined by* (4)*. Then* $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ *is an elementary solution of the operator* $\Diamond_B^k$*.*

The proof of this Lemma is given in [9].

Lemma 6. *Let k and r be nonnegative integer.*

(a) *Let $S_{2k}(x)$ and $S_{2r}(x)$ be defined by* (9), *then*

$$S_{2k}(x) * S_{2r}(x) = S_{2k+2r}(x).$$

(b) *Let $R_{2k}(x)$ and $R_{2r}(x)$ be defined by* (10), *then*

$$R_{2k}(x) * R_{2r}(x) = R_{2k+2r}(x).$$

The proof of this Lemma is given in [10].

Lemma 7. *The convolution* $S_{4k}(x) * R_{6k}(x)$ *exists and is a tempered distribution where* $S_{4k}(x) = S_{2k}(x) * S_{2k}(x)$ *and* $R_{6k}(x) = R_{2k}(x) * R_{2k}(x) * R_{2k}(x)$ *such that* $S_{2k}(x)$ *and* $R_{2k}(x)$ *are defined by* (9) *and* (10), *respectively.*

Proof. Since $S_{2k}(x) * R_{2k}(x)$ exists and is a tempered distribution, by Donoghue [1] page 156-159, we obtain $S_{4k}(x) * R_{6k}(x)$ exists and is a tempered distribution. □

Lemma 8. *Let* $S_4(x)$ *and* $R_6(x)$ *be defined by* (9) *and* (10) *with* $k = 2$ *and* $k = 3$*, respectively. Then*

(a) $\Diamond_B \triangle_B \left(S_4(x) * R_6(x)\right) = R_4(x)$,

(b) $\Box_B^3 \left(S_4(x) * R_6(x)\right) = S_4(x)$.

Proof. (a) We obtain

$$\begin{aligned}
&\Diamond_B \triangle_B \left(S_4(x) * R_6(x)\right) \\
&\qquad = \Diamond_B \left((-1)S_2(x) * R_2(x)\right) * \triangle_B \left((-1)S_2(x)\right) * R_4(x) \\
&\qquad = \delta(x) * \delta(x) * R_4(x) \\
&\qquad = R_4(x).
\end{aligned}$$

(b) We get

$$\begin{aligned}\Box_B^3\left(S_4(x) * R_6(x)\right) &= S_4(x) * \Box_B^3 R_6(x) \\ &= S_4(x) * \delta(x) \\ &= S_4(x).\end{aligned}$$

$\Box$

Main results

Theorem 9. *Given the equation*

$$\otimes_B^k G(x) = \delta(x), \tag{11}$$

then $G(x) = S_{4k}(x) * R_{6k}(x) * \left(C^{*k}(x)\right)^{*-1}$ *is a Green function for the operator* $\otimes_B^k$ *iterated* k*-times where* $\otimes_B$ *is defined by* (8), δ *is the Direc delta distribution,* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$, k *is a nonnegative integer and*

$$C(x) = \frac{3}{4} R_4(x) + \frac{1}{4} S_4(x), \tag{12}$$

$C^{*k}(x)$ *denotes the convolution of* C *itself* k*-times,* $\left(C^{*k}(x)\right)^{*-1}$ *denotes the inverse of* $C^{*k}(x)$ *in the convolution algebra. Moreover* $C^{*k}(x)$ *is a tempered distribution.*

Proof. Since $\otimes_B = \frac{3}{4}\Diamond_B\triangle_B + \frac{1}{4}\Box_B^3$, by (11) we obtain

$$\left[\frac{3}{4}\Diamond_B\triangle_B + \frac{1}{4}\Box_B^3\right]\left[\frac{3}{4}\Diamond_B\triangle_B + \frac{1}{4}\Box_B^3\right]^{k-1} G(x) = \delta(x).$$

By Lemma 7 with $k = 1$, $S_4(x) * R_6(x)$ exists and is a tempered distribution. Convolving both side of the above equation by $S_4(x) * R_6(x)$, we have

$$\left[\frac{3}{4}\Diamond_B\triangle_B + \frac{1}{4}\Box_B^3\right](S_4(x) * R_6(x)) * \left[\frac{3}{4}\Diamond_B\triangle_B + \frac{1}{4}\Box_B^3\right]^{k-1} G(x)$$
$$= (S_4(x) * R_6(x)) * \delta(x).$$

By Lemma 8, we obtain

$$C(x) * \left[\frac{3}{4}\Diamond_B\triangle_B + \frac{1}{4}\Box_B^3\right]^{k-1} G(x) = S_4(x) * R_6(x).$$

Keeping on convolving both sides of the above equation by $S_4(x) * R_6(x)$ up to $k-1$ times, we have

$$C^{*k}(x) * G(x) = \left(S_4(x) * R_6(x)\right)^{*k}.$$

where the symbol $*k$ denotes the convolution of itself k-times. By Tellez [5], we get

$$\left(S_4(x) * R_6(x)\right)^{*k} = S_{4k}(x) * R_{6k}(x).$$

Therefore,

$$C^{*k}(x) * G(x) = S_{4k}(x) * R_{6k}(x). \tag{13}$$

Since $S_4(x)$ and $R_4(x)$ are lies in S' where S' is a space of tempered distribution, $C(x) \in S'$. By Donoghue [1, p. 152], we obtain $C^{*k}(x) \in S'$. Since $S_{4k}(x) * R_{6k}(x) \in S'$, choose $S' \subset D'_R$ where D'_R is the right-side distribution which is a subspace of D' of distribution. Thus $S_{4k}(x) * R_{6k}(x) \in D'_R$, it follows that $S_{4k}(x) * R_{6k}(x)$ is an element of the convolution algebra.

By Zemanian [11, p. 150-151] the equation (13) has an unique solution

$$G(x) = S_{4k}(x) * R_{6k}(x) * \left(C^{*k}(x)\right)^{*-1}$$

where $\left(C^{*k}(x)\right)^{*-1}$ is an inverse of $C^{*k}(x)$ in the convolution algebra, $G(x)$ is called the elementary solution of the operator $\otimes_B^k$. Since $S_{4k}(x) * R_{6k}(x)$ and $\left(C^{*k}(x)\right)^{*-1}$ are tempered distribution, by Donoghue [1, p. 152] we obtain $S_{4k}(x) * R_{6k}(x) * \left(C^{*k}(x)\right)^{*-1}$ is a tempered distribution. It follows that $G(x)$ is a tempered distribution. $\square$

Theorem 10. *Given the equation*

$$\otimes_B^k u(x) = f(x) \tag{14}$$

where f is a given generalized function and $u(x)$ is an unknown function, we obtain

$$u(x) = G(x) * f(x)$$

is an unique solution of (14) *where $G(x)$ is an elementary solution for the operator $\otimes_B^k$.*

Proof. Convolving both sides of the equation (14) by Green function $G(x)$ for operator $\otimes_B^k$ in Theorem 9, we obtain

$$G(x) * \otimes_B^k u(x) = G(x) * f(x),$$

or

$$\otimes_B^k G(x) * u(x) = G(x) * f(x).$$

Applying Theorem 9, we have

$$\delta(x) * u(x) = G(x) * f(x) = u(x).$$

Since $G(x)$ is an unique, $u(x)$ is an unique solution of the equation (14). $\square$

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