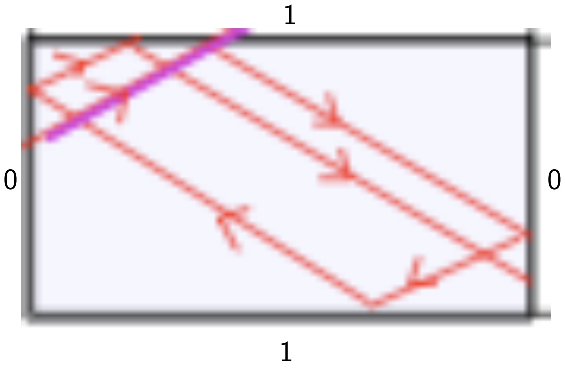
Billiards

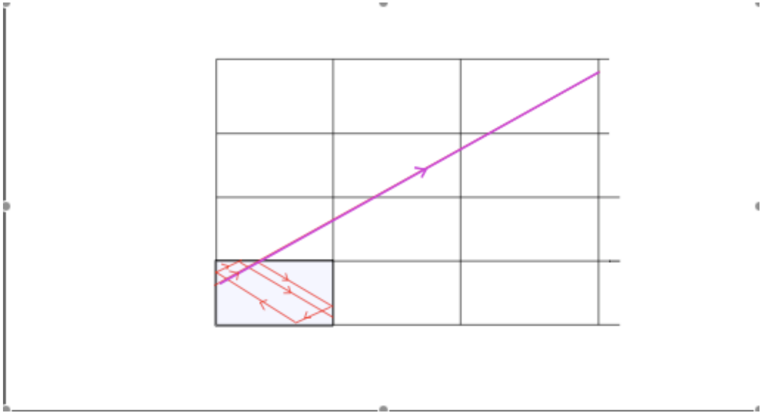
Pierre Arnoux

Bangkok, October 29, 2009

Billiards sequences: definition



Billiard sequence : unfolding



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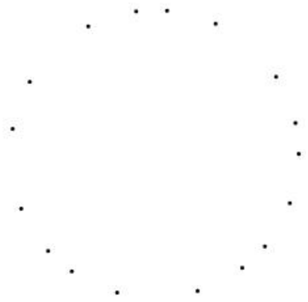

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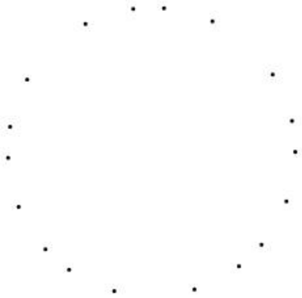

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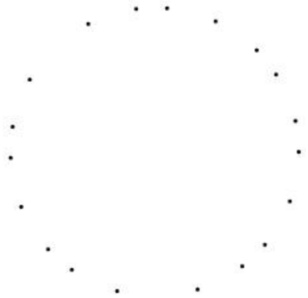

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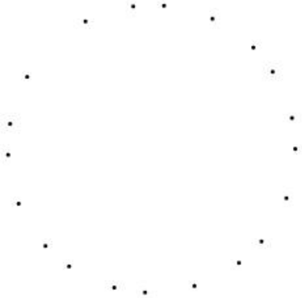

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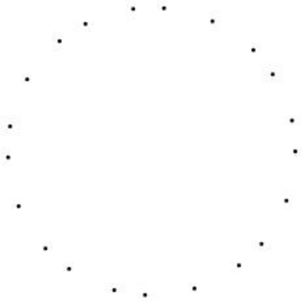

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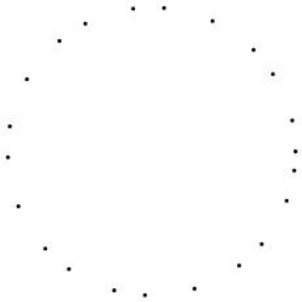

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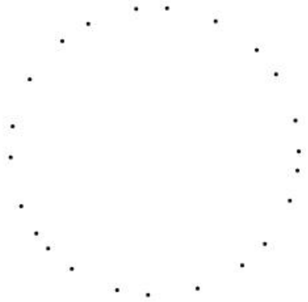

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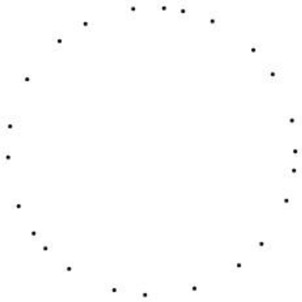

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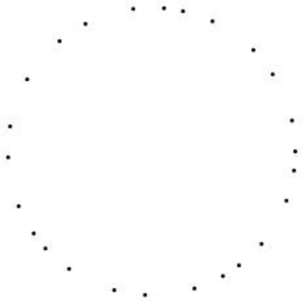

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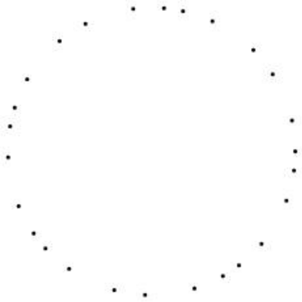

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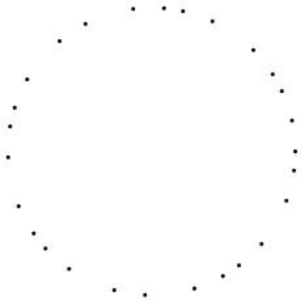

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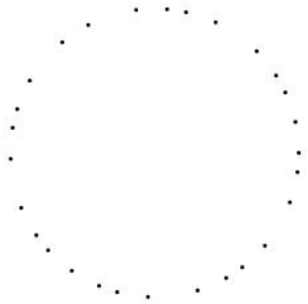

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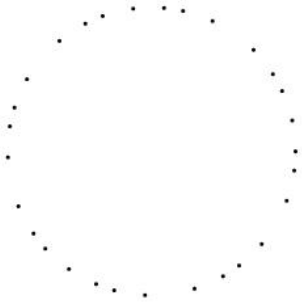

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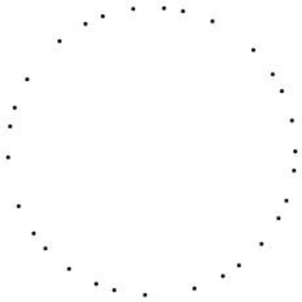

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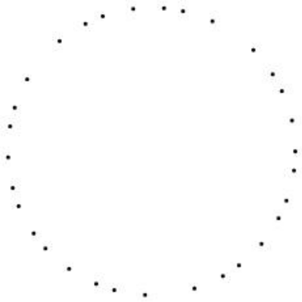

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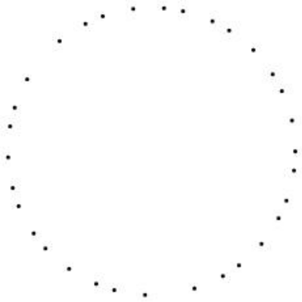

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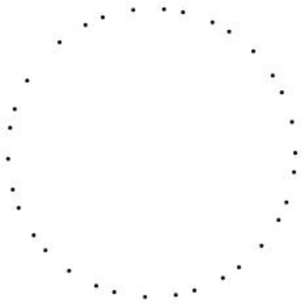

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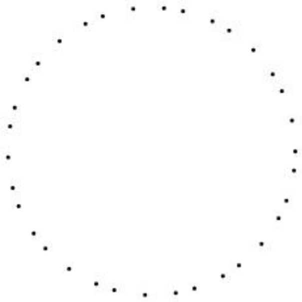

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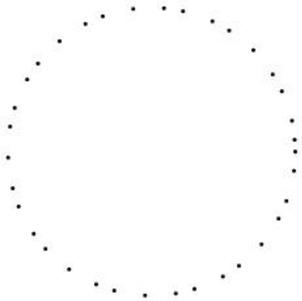

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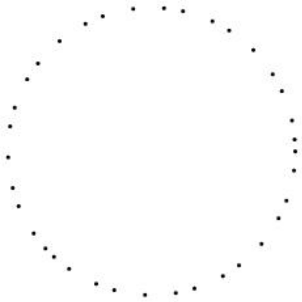

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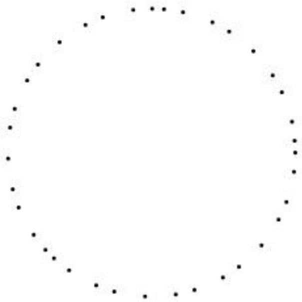

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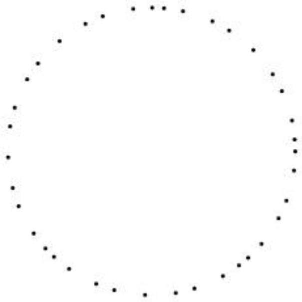

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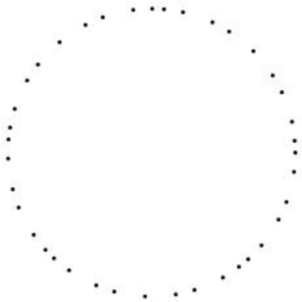

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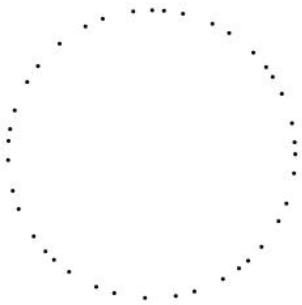

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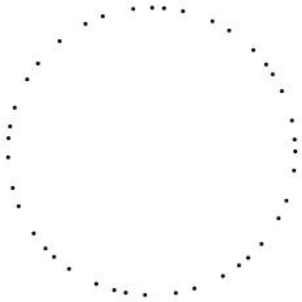

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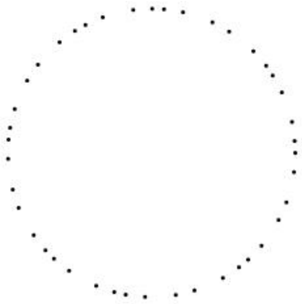

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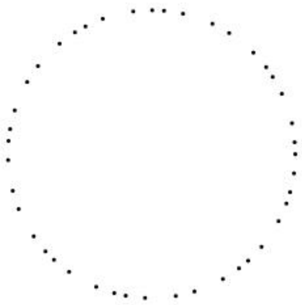

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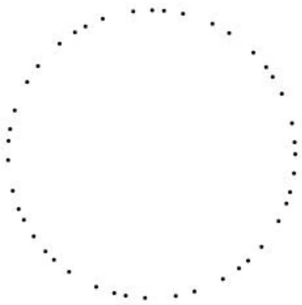

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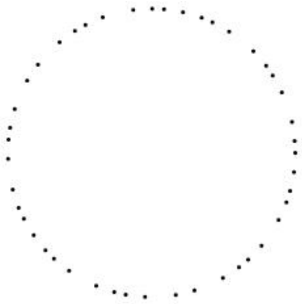

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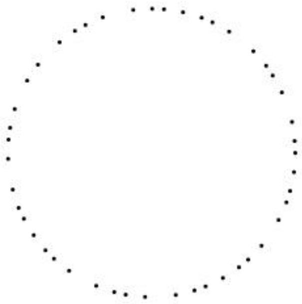

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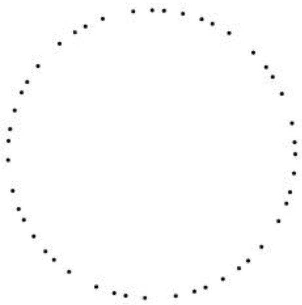

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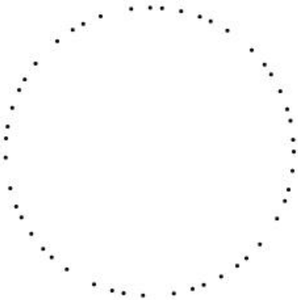

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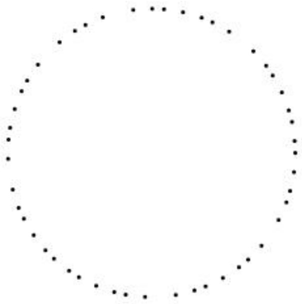

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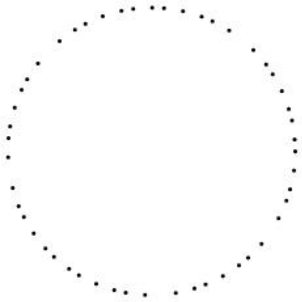

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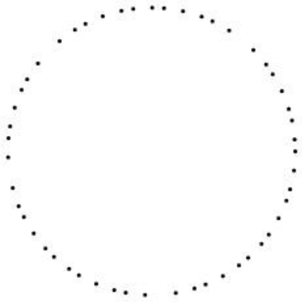

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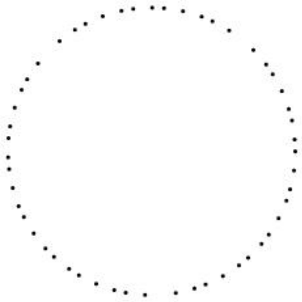

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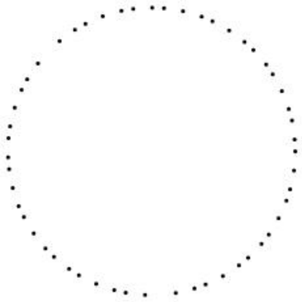

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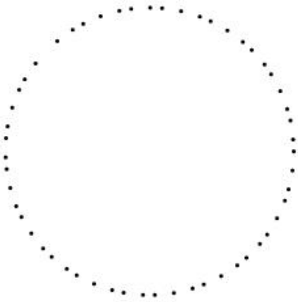

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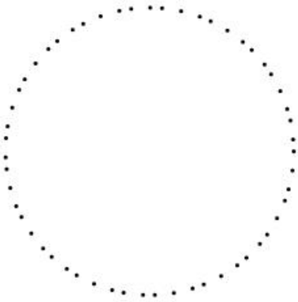

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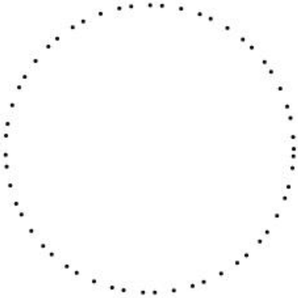

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- Unfolding: line in a grid
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- Occur every time there are two periodic phenomena with different period: calendar

Occur naturally in many contexts.
Many different characterizations.

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- 00 and 11 cannot occur simultaneously
- If 11 does not occur, we can erase the next 0
- The new sequence is again a rotation sequence
- This is the continued fraction expansion of the slope

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Rational triangular billiard

Billiard in a right triangle with one angle $\frac{\pi}{8}$

The group generated by reflexions on the sides has order 16

By identifying sides, we obtain a regular octogon

Opposite sides are identified

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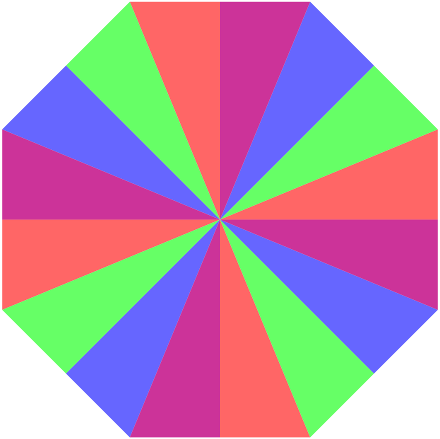
Octogonal Billiard



Octogonal sequences

We identify opposite sides, and follow a given direction

Recording the sides we cross, we obtain a symbolic octogonal sequence

This sequence is given by an exchange of 4 intervals

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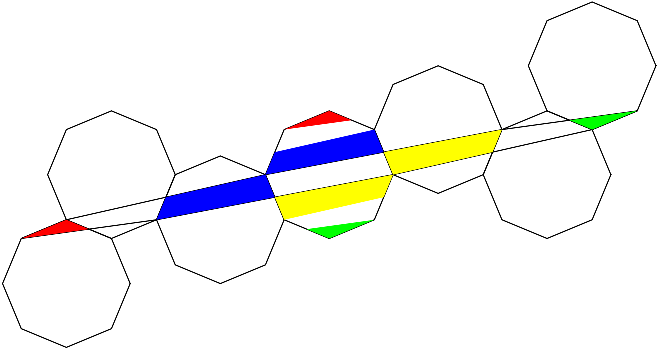
Renormalization of the octogon



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Due to Veech (1985)

In the framework of Teichmüller spaces

Definition of the Veech group V_4, conjugate to a subgroup of the Hecke group H_4

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Renormalization 2 : The Veech Continued fraction

The Veech group acts on the lines through the origin; one defines the Veech continued fraction, on the slope of these lines, in the following way:

Line with slope in $[-\frac{\mu}{2}, \frac{\mu}{2}]$.

Act by R_q till the slope gets out of the interval. Then act by the parabolic element P_q till the slope comes back to the interval.

One obtains in this way a generalized continued fraction, the *additive Veech Algorithm* (Arnoux-Hubert)

Question: how do we recover the octogon sequence?

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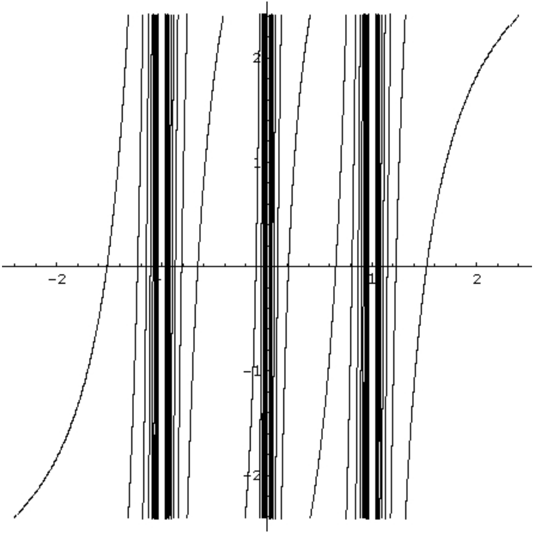

Renormalization 3 : The Ferenczi-Zamboni continued fraction

- Idea 1: consider exchanges of 4 intervals
- Idea 2: induce on a suitable set
- Idea 3: use it to recover special words for the symbolic dynamics, and even bi-special words
- This leads to a remarquable explicit algorithm for all exchanges of 4 intervals, which generalizes the usual continued fraction

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Renormalization 4: The Smillie-Ulcigrai continued fraction

Octogon sequences belong to one of 8 types (instead of 2 for rotations)

A letter is *sandwiched* if it is surrounded by identical letters : *...bab...*

The derived sequence of an admissible sequence is the subsequence of sandwiched letters

An admissible sequence is derivable if its derived sequence is admissible

Theorem: *Any octogon sequence is infinitely admissible*

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